Math. 401 (Fulling) 24 February 2006

Test A – Solutions

"Up **to** [a certain order]" *means* "stop right before calculating the term of that order."
"Up **through** [a certain order]" *means* "**do** calculate a term of that order."

Calculators may be used for simple arithmetic operations only!

1. *(32 pts.)* Consider $\epsilon x^3 + x - 1 = 0 \quad (\epsilon > 0,\ \epsilon \to 0)$.

(a) Find the leading behavior of all roots.

First let's see what regular perturbation theory produces.

$$x_0 - 1 = 0 \ \Rightarrow\ x_0 = 1.$$

There should be two more roots, so we try the balancing-and-scaling trick. If the ϵ term is to balance the highest-degree other term, then

$$\epsilon x^3 \approx -x \ \Rightarrow\ x \sim \epsilon^{-1/2}.$$

So we set $x = \epsilon^{-1/2}\overline{x}$, getting $\overline{x} \equiv \epsilon^{+1/2}x$ and therefore

$$\overline{x}^3 + \overline{x} - \sqrt{\epsilon} = 0.$$

The zeroth-order equation is then

$$0 = \overline{x}_0{}^3 + \overline{x}_0 = \overline{x}_0(\overline{x}_0{}^2 + 1) \ \Rightarrow\ \overline{x}_0 = \pm i$$

(since the root $\overline{x}_0 = 0$ would merely reproduce the regular root). Thus

$$x_0 = \frac{\pm i}{\sqrt{\epsilon}}\,.$$

(b) For all **real** roots, find a perturbation expansion through order ϵ .

The only real root is the regular one. Make the ansatz $x \sim 1 + \epsilon x_1$ (because we already know that $x_0 = 1$). Then $x^3 \sim 1 + 3\epsilon x_1 + O(\epsilon^2)$ and hence the equation is

$$\begin{aligned} 0 &\sim \epsilon + 3\epsilon^2 x_1 + \cdots + 1 + \epsilon x_1 - 1 \\ &= \epsilon(1 + x_1) + O(\epsilon^2). \end{aligned}$$

Thus $x_1 = -1$, and

$$x \sim 1 - \epsilon.$$

2. *(24 pts.)* Do **ONE** of these [(A) **or** (B)].

(A) Write down the (first-order) WKB approximation to the solution of

$$\frac{d^2y}{dt^2} + [\omega^2 - t^2]y = 0, \quad \omega \to +\infty,$$

with initial data $y(0) = 1$, $y'(0) = 0$. (You are not expected to evaluate the integral that arises.) Then discuss the uniformity (or lack thereof) of the approximation. (Can the solution be extended to the whole domain $-\infty < t < \infty$? If not, why not?)

The ODE has two linearly independent solutions with WKB approximations

$$y_\pm \sim [\omega^2 - t^2]^{-1/4} e^{\pm i \int_0^t \sqrt{\omega^2 - \tilde{t}^2}\, d\tilde{t}}$$

(see p. 44 of class notes). Their derivatives are

$$y'_\pm \sim \pm i[\omega^2 - t^2]^{+1/4} e^{\pm i \int_0^t \sqrt{\omega^2 - \tilde{t}^2}\, d\tilde{t}} - \frac{1}{4}[\omega^2 - t^2]^{-5/4}(-2t) e^{\pm i \int_0^t \sqrt{\omega^2 - \tilde{t}^2}\, d\tilde{t}}.$$

The last term of this expression is actually of higher order than the first one, so it can be dropped in constructing the first-order approximation; but since this point is not obvious, I will carry the term along for awhile. Consider $y = c_+ y_+ + c_- y_-$ and impose the initial conditions:

$$1 = c_+\omega^{-1/2} + c_-\omega^{-1/2}, \qquad 0 = c_+ i\omega^{1/2} - c_- i\omega^{1/2} + O\big(\omega^{-5/2}\big).$$

(In fact, the $O\big(\omega^{-5/2}\big)$ term is zero in this problem because of the factor $-2t$, but for a potential with $V'(0) \neq 0$ it would be nonzero but negligible.) Rewrite the system as

$$c_+ + c_- = \omega^{+1/2}, \qquad c_+ - c_- = 0.$$

The solution is

$$c_\pm = \frac{1}{2}\omega^{1/2}.$$

Thus

$$y(t) \sim \frac{1}{2}\omega^{1/2}[\omega^2 - t^2]^{-1/4}\left[e^{i \int_0^t \sqrt{\omega^2 - \tilde{t}^2}\, d\tilde{t}} + e^{-i \int_0^t \sqrt{\omega^2 - \tilde{t}^2}\, d\tilde{t}}\right].$$

(It could also be written as a hyperbolic cosine.) This approximation is valid in a neighborhood of $t = 0$, but it must go bad when t becomes close to $\pm\omega$, and it is completely wrong when $\omega^2 - t^2$ is zero or negative. How large the "good" interval is depends on ω; it can be made arbitrarily large by considering only very large ω, but it is small for small ω. In other words, for fixed ω the approximation is not uniform in t, and for fixed t the size of the interval where the approximation is useful is not uniform in ω.

(B) Pronounce each of the following assertions true or false, and write something to explain your judgment.

(a) $e^{\epsilon t} - 1 = O(\epsilon)$ as $\epsilon \to 0$ (for fixed t).

True. This is just the statement that the exponential function has a Taylor series around $\epsilon = 0$ with leading term 1.

(b) $\epsilon^3 \cosh\epsilon = O(\epsilon^4)$ as $\epsilon \to 0$.

False. The Taylor series of $\cosh\epsilon \equiv \frac{1}{2}(e^{\epsilon} + e^{-\epsilon})$ is $1 + \frac{1}{2}\epsilon^2 + \cdots$. So our function is $O(\epsilon^3)$ but not $O(\epsilon^4)$.

(c) $\dfrac{\sin\epsilon}{\sqrt{1-\epsilon}} = O\bigl(\epsilon^{3/2}\bigr)$ as $\epsilon \to 0$.

False. The leading behavior of $\sin\epsilon$ is ϵ, while the leading behavior of the denominator is 1. (More precisely, $(1-\epsilon)^{-1/2} \sim 1 + \frac{1}{2}\epsilon$.) So the function is $O(\epsilon)$ but not $O\bigl(\epsilon^{3/2}\bigr)$.)

(d) $\ln\epsilon = O\bigl(\epsilon^{-1}\bigr)$ as $\epsilon \to 0$ (with $\epsilon > 0$).

True. At 0 the logarithm function goes to infinity more slowly than any power. In more detail: The statement is equivalent to

$$\frac{\ln\epsilon}{\epsilon^{-1}} \quad \text{is bounded for } \epsilon \text{ near zero.}$$

It is easier to prove the stronger statement ["$\ln\epsilon = o(\epsilon^{-1})$"] that

$$0 = \lim_{\epsilon\downarrow 0} \frac{\ln\epsilon}{\epsilon^{-1}} = \lim_{\epsilon\downarrow 0} \epsilon\ln\epsilon,$$

and that is easily verified by l'Hôpital's rule:

$$\lim_{\epsilon\downarrow 0} \frac{\ln\epsilon}{\epsilon^{-1}} = \lim_{\epsilon\downarrow 0} \frac{1/\epsilon}{-\epsilon^{-2}} = -\lim_{\epsilon\downarrow 0} \epsilon = 0.$$

3. *(44 pts.)* Find a two-term solution (through order ϵ) by the distorted-time method:

$$\frac{d^2y}{dt^2} + \frac{36}{\sqrt{\epsilon}}\sin(\sqrt{\epsilon}\,y) = 0, \quad y(0) = 0, \quad \frac{dy}{dt}(0) = 6 \qquad (\epsilon > 0,\ \epsilon \to 0).$$

You may find some of the following formulas helpful.

$$\sin x \sim x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots, \qquad \cos x \sim 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

$$\cos^2 x = \tfrac{1}{2} + \tfrac{1}{2}\cos(2x), \qquad \sin^2 x = \tfrac{1}{2} - \tfrac{1}{2}\cos(2x)$$

$$\sin x\cos y = \tfrac{1}{2}\sin(x-y) + \tfrac{1}{2}\sin(x+y)$$

$$\cos x\cos y = \tfrac{1}{2}\cos(x-y) + \tfrac{1}{2}\cos(x+y)$$

$$\sin x\sin y = \tfrac{1}{2}\cos(x-y) - \tfrac{1}{2}\cos(x+y)$$

$$\cos^3 x = \tfrac{1}{4}\cos(3x) + \tfrac{3}{4}\cos x, \qquad \sin^3 x = -\tfrac{1}{4}\sin(3x) + \tfrac{3}{4}\sin x$$

$$\cos^2 x\sin x = \tfrac{1}{4}\sin(3x) + \tfrac{1}{4}\sin x, \qquad \cos x\sin^2 x = -\tfrac{1}{4}\cos(3x) + \tfrac{1}{4}\cos x$$

Expand the nonlinear term:

$$\sin(\sqrt{\epsilon}\,y) \sim \sqrt{\epsilon}\,y - \frac{1}{6}\,(\sqrt{\epsilon}\,y)^3 + O\big(\epsilon^{5/2}\big),$$

so the ODE is

$$\frac{d^2y}{dt^2} + 36y - 6\epsilon y^3 \sim 0.$$

Introduce a new time scale by $\tau = (1+\epsilon\omega_1)t$, so that

$$\frac{d}{dt} = (1+\epsilon\omega_1)\,\frac{d}{d\tau}, \quad \frac{d^2}{dt^2} = (1+2\epsilon\omega_1 + O(\epsilon^2))\,\frac{d^2}{d\tau^2}\,.$$

Thus the problem becomes

$$\frac{d^2y}{d\tau^2} + 36y + 2\epsilon\omega_1\frac{d^2y}{d\tau^2} - 6\epsilon y^3 \sim 0, \qquad y(0)=0, \quad (1+\epsilon\omega_1)\,\frac{dy}{d\tau}(0) = 6.$$

Now let $y \sim y_0 + \epsilon y_1$ and group terms of the same order:

ϵ^0 :
$$\frac{d^2y_0}{d\tau^2} + 36y_0 = 0, \qquad y_0(0) = 0, \quad \frac{dy_0}{d\tau}(0) = 6.$$

ϵ^1 :
$$\frac{d^2y_1}{d\tau^2} + 36y_1 = -2\omega_1\frac{d^2y_0}{d\tau^2} + 6{y_0}^3, \qquad y_1(0)=0, \quad \frac{dy_1}{d\tau}(0) = -\omega_1\frac{dy_0}{d\tau}(0).$$

The ϵ^0 term is quickly seen to be

$$y_0 = \sin(6\tau),$$

so the ϵ^1 equation is

$$\begin{aligned}\frac{d^2y_1}{d\tau^2} + 36y_1 &= +72\omega_1\sin(6\tau) + 6\sin^3(6\tau)\\ &= 72\omega_1\sin(6\tau) - \frac{3}{2}\,\sin(18\tau) + \frac{9}{2}\,\sin(6\tau).\end{aligned}$$

To avoid a secular term we must take $72\omega_1 + \frac{9}{2} = 0$, or

$$\omega_1 = -\,\frac{1}{16}\,.$$

Now the ϵ^1 problem is

$$\frac{d^2y_1}{d\tau^2} + 36y_1 = -\frac{3}{2}\,\sin(18\tau), \qquad y_1(0) = 0, \quad \frac{dy_1}{d\tau}(0) = \frac{1}{16}\,\frac{dy_0}{d\tau}(0) = \frac{3}{8}\,.$$

Look for a particular solution $y_p = A\sin(18\tau)$:

$$\frac{d^2y_p}{d\tau^2} + 36y_p = (-18^2 + 36)y_p = -\frac{3}{2}\sin(18\tau),$$

or

$$A = \frac{-3/2}{18(-18+2)} = \frac{1}{192}\,.$$

Then

$$y_1 = c_1\cos(6\tau) + c_2\sin(6\tau) + \frac{1}{192}\sin(18\tau),$$

whence, from the initial conditions, $0 = c_1$ and

$$\frac{3}{8} = 6c_2 + \frac{18}{192} = 6c_2 + \frac{3}{32}\,,$$

so that

$$c_2 = \frac{3}{6}\left(\frac{1}{8} - \frac{1}{32}\right) = \frac{3}{64}\,.$$

So, finally,

$$y(t) \sim \sin(6\tau) + \epsilon\left[\frac{3}{64}\sin(6\tau) + \frac{1}{192}\sin(18\tau)\right]$$

where

$$\tau = t - \frac{\epsilon}{16}\,t.$$