Math. 401 (Fulling) 11 May 2009

Final Examination – Solutions

Calculators may be used for simple arithmetic operations only!

Possibly useful formulas

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\, e^{-k^2 t}\, dk = \frac{1}{\sqrt{4\pi t}}\, e^{-x^2/4t}\,, \qquad \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\, e^{-|k|y}\, dk = \frac{1}{\pi}\,\frac{y}{x^2+y^2}\,.$$

$$\cos^3 x = \tfrac{1}{4}\cos(3x) + \tfrac{3}{4}\cos x\,, \qquad \sin^3 x = -\tfrac{1}{4}\sin(3x) + \tfrac{3}{4}\sin x\,,$$

$$\cos^2 x \sin x = \tfrac{1}{4}\sin(3x) + \tfrac{1}{4}\sin x\,, \qquad \cos x \sin^2 x = -\tfrac{1}{4}\cos(3x) + \tfrac{1}{4}\cos x\,.$$

$$\sinh a \cosh b + \sinh b \cosh a = \sinh(a+b)\,.$$

1. *(30 pts.)* Classify each equation as elliptic, hyperbolic, or parabolic, **and** as nonlinear, linear homogeneous, or linear nonhomogeneous.

(a) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - 2\,\frac{\partial u}{\partial y} + y^2 = 0\,.$$

elliptic, linear nonhomogeneous

(b) $$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2} = u^2\,.$$

parabolic ($B^2 - 4AC = 4 - 4 = 0$), nonlinear

(c) $$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 2u\,.$$

hyperbolic, linear homogeneous

2. *(40 pts.)* Solve (by the method of your choice)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad (0 < x < \infty\,, \quad 0 < t < \infty)\,,$$

$$u(t,0) = 0\,, \qquad u(0,x) = f(x)\,.$$

Simplify your answer as much as possible (*hint:* Green function).

Method 1: A Fourier sine transform in x is indicated.

$$u(t,x) = \int_0^\infty B(k)\sin(kx)e^{-k^2 t}\, dk$$

where $f(x) = \int_0^\infty B(k)\sin(kx)\,dk$ and hence

$$B(k) = \frac{2}{\pi}\int_0^\infty f(x)\sin(kx)\,dx.$$

Thus (after changing x to z in the integral)

$$u(t,x) = \frac{2}{\pi}\int_0^\infty dz \int_0^\infty dx\,\sin(kx)\sin(kz)e^{-k^2 t} f(z) = \int_0^\infty H(t,x,z) f(z)\,dz$$

where

$$\begin{aligned} H(t,x,z) &= -\frac{2}{\pi}\frac{1}{4}\int_0^\infty dk\,\left(e^{kx} - e^{-kx}\right)\left(e^{kz} - e^{-kz}\right)e^{-k^2 t} \\ &= -\frac{1}{2\pi}\int_{-\infty}^\infty dk\,\left(e^{k(x+z)} - e^{k(x-z)}\right)e^{-k^2 t} \\ &= \frac{1}{\sqrt{4\pi t}}\left(e^{-(x-z)^2/4t} - e^{-(x+z)^2/4t}\right). \end{aligned}$$

(The first "possibly useful formula" was used at the last step.)

Method 2: The Green function for the heat equation on the whole real line is well known to be

$$H_0(t,x,z) = \frac{1}{\sqrt{4\pi t}}\,e^{-(x-z)^2/4t}.$$

The solution on the half-line with the Dirichlet condition can be obtained by considering the odd extension of f to the whole real line, so that

$$u(t,x) = \int_{-\infty}^\infty H_0(t,x,z) f(z)\,dz\,,$$

and then using the oddness of f to get the "image solution"

$$u(t,x) = \int_0^\infty \left(H_0(t,x,z) - H_0(t,x,-z)\right) f(z)\,dz\,.$$

This is the same solution as above.

3. *(40 pts.)* Find the eigenvalues and orthonormal eigenfunctions of this two-dimensional problem (which might arise in solving the heat equation in a cylindrical pipe):

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\omega^2 u \qquad (-\pi < x < \pi\,, \quad 0 < y < L)\,,$$

periodic boundary conditions in x,

$$\frac{\partial u}{\partial y}(x,0) = 0\,, \qquad u(x,L) = 0\,.$$

If we separate variables as $u = X(x)Y(y)$, we get

$$X'' = -\mu^2 X\,, \quad Y'' = -\nu^2 Y \quad \text{with } \omega^2 = \mu^2 + \nu^2\,.$$

Because the interval length in the periodic direction is the standard one, 2π , we have

$$X_m(x) = e^{imx}\,, \quad \mu_m = m\,, \quad m \in \mathbf{Z}\,.$$

In the y direction we have $Y = \cos(\nu y)$ with $0 = Y(L) = \cos(\nu L)$, thus νL an odd multiple of $\pi/2$:

$$Y_n(y) = \cos(\nu_n y)\,, \quad \nu_n = \left(n - \frac{1}{2}\right)\frac{\pi}{L}\,, \quad n = 0, 1, \ldots, \infty\,.$$

The eigenfunctions thus form a two-index family of products of these one-dimensional eigenfunctions.

To get the normalization right, integrate $[X_m(x)Y_n(y)]^2$ over the variable range $-\pi < x < \pi$, $0 < y < L$, getting

$$2\pi \int_0^L \cos^2(\nu_n y)\,dy = \pi L\,.$$

To make the functions orthonormal, we must divide by the square root of this number. Thus each normalized eigenfunction has the form

$$\Phi_{mn}(x,y) = \frac{1}{\sqrt{\pi L}}\, e^{imx} \cos(\nu_n y)\,, \qquad \nu_n \equiv \left(n - \frac{1}{2}\right)\frac{\pi}{L}\,.$$

Equivalently, include from the beginning the known normalization factor for each of the one-dimensional problems:

$$\frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{L}} = \frac{1}{\sqrt{\pi L}}\,.$$

4. *(40 pts.)* Apply the method of two time scales to

$$\frac{d^2 y}{dt^2} + \epsilon\left(\frac{dy}{dt}\right)^3 + y = 0\,, \qquad y(0) = 1\,, \quad y'(0) = 0\,.$$

(Stop after you have found the coupled first-order differential equations satisfied by the coefficients in the leading term.)

Let $\tau = \epsilon t$ and $u = u(t,\tau)$, so that $\dfrac{d}{dt} = \dfrac{\partial}{\partial t} + \epsilon\,\dfrac{\partial}{\partial \tau}$. Through order ϵ , the equation is

$$\frac{\partial^2 y}{\partial t^2} + 2\epsilon\,\frac{\partial^2 y}{\partial t\,\partial \tau} + \epsilon\left(\frac{\partial y}{\partial t}\right)^3 + y = 0\,.$$

If $y = y_0 + \epsilon y_1$, then,

$$0 = \frac{\partial^2 y_0}{\partial t^2} + \epsilon\,\frac{\partial^2 y_1}{\partial t^2} + 2\epsilon\,\frac{\partial^2 y_0}{\partial t\,\partial \tau} + \epsilon\left(\frac{\partial y_0}{\partial t}\right)^3 + y_0 + \epsilon y_1\,.$$

The initial conditions are (through order ϵ)

$$1 = y_0(0,0) + \epsilon y_1(0,0)\,,$$

$$0 = \left(\frac{\partial}{\partial t} + \epsilon\,\frac{\partial}{\partial \tau}\right)(y_0 + \epsilon y_1) = \frac{\partial y_0}{\partial t} + \epsilon\,\frac{\partial y_0}{\partial \tau} + \epsilon\,\frac{\partial y_1}{\partial t} \quad \text{at } (0,0)\,.$$

Order ϵ^0 :

$$0 = \frac{\partial^2 y_0}{\partial t^2} + y_0\,, \qquad y_0(0,0) = 1\,, \quad \frac{\partial y_0}{\partial t}(0,0) = 0\,.$$

Thus $y_0 = A(\tau)\cos t + B(\tau)\sin t$ where $A(0) = 1$ and $B(0) = 0$.

Order ϵ^1 :

$$0 = \frac{\partial^2 y_1}{\partial t^2} + y_1 + 2\,\frac{\partial^2 y_0}{\partial t\,\partial \tau} + \left(\frac{\partial y_0}{\partial t}\right)^3\,.$$

$$\begin{aligned}
\frac{\partial^2 y_1}{\partial t^2} + y_1 &= 2A'(\tau)\sin t - 2B'(\tau)\cos t + A^3\sin^3 t - 3A^2B\sin^2 t\cos t + 3AB^2\sin t\cos^2 t - B^3\cos^3 t \\
&= 2A'\sin t - 2B'\cos t + \frac{A^3}{4}[-\sin(3t) + 3\sin t] \\
&\quad - \frac{3A^2B}{4}[-\cos(3t) + \cos t] + \frac{3AB^2}{4}[\sin(3t) + \sin t] - \frac{B^3}{4}[\cos(3t) + 3\cos t] \\
&= \text{nonresonant terms} + \left[2A' + \frac{3A^3}{4} + \frac{3AB^2}{4}\right]\sin t + \left[-2B' - \frac{3A^2B}{4} - \frac{3B^3}{4}\right]\cos t\,.
\end{aligned}$$

So the equations that must be solved are

$$A' = -\frac{3}{8}(A^3 + AB^2)\,, \qquad B' = -\frac{3}{8}(A^2B + B^3)\,,$$

with $A(0) = 1$ and $B(0) = 0$.

5. *(40 pts.)* Do **ONE** of these (A, B. C, **or** D). **(NO extra credit for doing more than one. Indicate which one you want graded!)**

(A) Construct the Green function for the problem

$$\frac{d^2u}{dx^2} - u = f(x)\,, \qquad u(0) = 0\,, \quad u(1) = 0\,.$$

That is, express the solution in the form

$$u(x) = \int_0^1 G(x,y) f(y)\,dy\,.$$

Solve

$$\frac{\partial^2 G}{\partial x^2} - G = \delta(x-y)\,.$$

From the homogeneous equation and the boundary conditions we get

$$G(x,y) = \begin{cases} A\sinh x & \text{for } x < y, \\ B\sinh(1-x) & \text{for } x > y. \end{cases}$$

Continuity requires

$$A\sinh y = B\sinh(1-y)\,. \tag{1}$$

The jump condition is

$$1 = \frac{\partial G}{\partial x}\bigg|_{y-\epsilon}^{y+\epsilon} = -B\cosh(1-y) - A\cosh y\,. \tag{2}$$

From (1) we get

$$A = B\,\frac{\sinh(1-y)}{\sinh y}$$

and hence from (2)

$$1 = -B\,\frac{\sinh y\cosh(1-y) + \cosh y\sinh(1-y)}{\sinh y}\,.$$

The numerator simplifies (by one of the hint formulas) to $\sinh 1$. Thus

$$A = -\frac{\sinh(1-y)}{\sinh 1}\,, \qquad B = -\frac{\sinh y}{\sinh 1}\,.$$

The final formula is best expressed by defining

$$x_< \equiv \min(x,y)\,, \qquad x_> \equiv \max(x,y)\,,$$

so that

$$G(x,y) = -\frac{\sinh(x_<)\sinh(1-x_>)}{\sinh 1}\,.$$

(B) Solve (by the method of your choice) the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad (0 < x < \infty\,, \quad -\infty < t < \infty)$$

with boundary condition

$$u(t,0) = 0$$

and initial conditions

$$u(0,x) = f(x)\,, \quad \frac{\partial u}{\partial t}(0,x) = g(x)\,.$$

Method 1: Fourier sine transform similar to Qu. 2. For each normal mode there are two solutions to the equation for the time dependence, and hence both initial conditions are needed to find all the coefficients.

Method 2: D'Alembert's solution. Henceforth interpret f and g as the *odd* extensions of the data functions to the whole real line. Let $G(x)$ be any antiderivative of g. Then

$$\begin{aligned} u(t,x) &= \frac{1}{2}[f(x+t) + f(x-t)] + \frac{1}{2}[G(x+t) - G(x-t)] \\ &= \frac{1}{2}[f(x+t) + f(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} g(z)\,dz\,. \end{aligned}$$

(C) Find the eigenfunctions and (graphically) the eigenvalues for the problem

$$X''(x) = -\omega^2 X(x)\,, \qquad X'(0) = X(0)\,, \quad X'(1) = 0\,.$$

Give an algebraic approximation for ω_n for n large. (You can assume there are no negative or zero eigenvalues.)

We must have $X(x) = A\cos(\omega x) + B\sin(\omega x)$. Thus $X'(x) = -A\omega\sin(\omega x) + B\omega\cos(\omega x)$. The boundary conditions then yield

$$B\omega = A \quad \text{and} \quad 0 = -A\omega\sin(\omega) + B\omega\cos(\omega)\,.$$

The allowed values of ω are those for which the determinant of this system vanishes:

$$0 = \begin{vmatrix} 1 & -\omega \\ -\omega\sin(\omega) & \omega\cos(\omega) \end{vmatrix} = \omega\cos(\omega) - \omega^2\sin(\omega)\,,$$

or

$$\omega = \cot(\omega)\,. \tag{3}$$

Sketch both sides of (3) on the same graph; the intersections are the square roots of the eigenvalues. For large n, ω_n is close to (slightly greater than) $n\pi$, the vertical asymptote of the cotangent function.

Alternate method: One boundary condition shows that $X(x) = C\cos[\omega(1-x)]$. Then (3) is obtained immediately from the other boundary condition.

(D) Use a Parseval equation to evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+y^2)^2}$.

We need to force the problem into the mold of $\int |f(x)|^2\,dx = \int |\hat f(k)|^2\,dk$. Rewrite the second "possibly useful formula" as

$$\begin{aligned} f(x) \equiv \frac{1}{x^2+y^2} &= \frac{1}{2y}\int_{-\infty}^{\infty} e^{ikx} e^{-|k|y}\,dk \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\left(\sqrt{\frac{\pi}{2}}\,\frac{1}{y}\right) e^{-|k|y}\,dk\,. \end{aligned}$$

So we can identify

$$\hat f(k) = \sqrt{\frac{\pi}{2}}\,\frac{1}{y}\,e^{-|k|y}\,.$$

Therefore,

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \frac{\pi}{2y^2}\int_{-\infty}^{\infty} e^{-2|k|y}\,dk = \frac{\pi}{y^2}\int_0^{\infty} e^{-2ky}\,dk = \frac{\pi}{2y^3}\,.$$

The last 10 points are free!