Math. 401, Sec. 501 Spring, 2009

Homework 10, due April 8

1. *[Cf. remark on p. 84 of notes.]* Find the Parseval equation for the Fourier cosine series.

   (a) Use the definition of the Fourier cosine series on pp. 78–79 of class notes.

   (b) How would your answer change if you used the other convention (p. 81 of notes)?

2. *[Schaum's, p. 46, Ex. 2.56 and 2.57]*

   (a) A square plate of side L has one side maintained at temperature $f(x)$ and the others at zero. Find the steady-state temperature at any point of the plate (as a Fourier series of appropriate type).

   (b) Explain how to solve the problem if the four sides are maintained at temperatures $f_1(x)$, $g_1(y)$, $f_2(x)$, and $g_2(y)$. (Write out the answer in full for the case $f_2 = 0 = g_2$.)

3. *[Schaum's, p. 46, Ex. 2.58(a)]* An infinitely long plate of width L has its two parallel sides maintained at temperature 0 and its other side at constant temperature T. Find the steady-state temperature.

4. *[Schaum's, p. 46, Ex. 2.63]* Solve the boundary-value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \alpha^2 u, \qquad u(0,t) = u_1\,, \quad u(L,t) = u_2\,, \quad u(x,0) = 0,$$

   where $0 < x < L$, $0 < t$, and α, L, u_1, and u_2 are constants.

5. *[Schaum's, p. 94, Ex. 5.42]* An infinite thin bar ($-\infty < x < \infty$) whose surface is insulated has an initial temperature

$$f(x) = \begin{cases} u_0 & \text{if } |x| < a, \\ 0 & \text{if } |x| \ge a. \end{cases}$$

   (a) Solve the heat equation $\dfrac{\partial u}{\partial t} = \kappa \dfrac{\partial^2 u}{\partial x^2}$ by separation of variables, obtaining a double integral.

   (b) Exchange the order of integration and evaluate the inner integral as the heat equation Green function (called "Gauss–Weierstrass kernel" by Constanda). Work the answer into the final form

$$u(x,t) = \frac{u_0}{2}\left[\operatorname{erf}\left(\frac{x+a}{2\sqrt{\kappa t}}\right) - \operatorname{erf}\left(\frac{x-a}{2\sqrt{\kappa t}}\right)\right].$$

   The *error function* erf is defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\, du.$$

6. *[Schaum's, p. 94, Ex. 5.43]* A semiinfinite solid ($x > 0$) has initial temperature $f(x) = u_0 e^{-bx^2}$. The plane face, $x = 0$, is insulated. Show that the temperature is

$$u(x,t) = \frac{u_0}{\sqrt{1 + 4\kappa bt}}\, e^{-bx^2/(1+4\kappa bt)}.$$

(Follow the same two steps as in the previous problem.)

7. *[Schaum's, p. 94, Ex. 5.46]* Solve the potential problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

for $y > 0$ with the boundary data

$$u(x,0) = \begin{cases} 0 & \text{if } x < -1 \text{ or } x > 1, \\ 1 & \text{if } -1 < x < 1. \end{cases}$$

8. *[Schaum's, p. 94, Ex. 5.49]* The lines $y = 0$ and $y = a$ in the xy plane are kept at potentials 0 and $f(x)$ respectively. In the strip between, the potential $u(x,y)$ obeys Laplace's equation (as in the previous problem). Show that the solution is

$$u(x,y) = \frac{1}{\pi} \int_{\lambda=0}^{\infty} \int_{\zeta=-\infty}^{\infty} f(\zeta)\, \frac{\sinh \lambda y}{\sinh \lambda a} \cos \lambda(\zeta - x)\, d\zeta\, d\lambda.$$