Math. 401, Sec. 501 — Spring, 2009

Homework 11, due April 20

1. *[Schaum's, p. 94, Ex. 5.24]*

   (a) Find the Fourier transform of $f(x) = \begin{cases} \dfrac{1}{2\epsilon} & \text{if } |x| < \epsilon, \\ 0 & \text{if } |x| \geq \epsilon. \end{cases}$

   (b) Find the limit of this transform as $\epsilon \to 0^+$. Discuss the result.

2. *[Schaum's, p. 94–95, Ex. 5.26 and 5.31]* Find the Fourier sine transform and the Fourier cosine transform of

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x < 1, \\ 0 & \text{if } x \geq 1. \end{cases}$$

   Use the results to show that

   (a) $\displaystyle\int_0^\infty \left(\frac{1 - \cos x}{x}\right)^2 dx = \frac{\pi}{2}$

   (b) $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2}\, dx = \frac{\pi}{2}$

3. *[Schaum's, p. 94, Ex. 5.37]* Find $y(x)$ given that

$$\int_{-\infty}^{\infty} y(u) y(x-u)\, du = e^{-x^2}.$$

4. *[Schaum's, p. 94, Ex. 5.39]* Prove that $f * (g * h) = (f * g) * h$.

5. Show that differentiation of a Fourier transform with respect to ω corresponds to multiplication of the original function by $-ix$,

   (a) starting from the equation

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x}\, \hat{g}(\omega)\, d\omega,$$

   applied to $\hat{g}(\omega) = \hat{f}\,'(\omega)$;

   (b) alternatively, starting from the equation

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x}\, f(x)\, dx.$$

   Note: With Constanda's definition of the Fourier transform one would get $+ix$ instead of $-ix$.

6. Starting from the Fourier transform formulas in exponential form (see previous problem, with $g = f$), derive the formulas for the Fourier sine transform. (Apply the Fourier transform formulas to the odd extension of f, f being an arbitrary nice function defined on $[0, \infty)$.)

7. *[Schaum's, p. 94, Ex. 5.27]*

   (a) Find the Fourier sine transform of e^{-x} $(x \geq 0)$.

   (b) Use (a) to show that for $m > 0$,

   $$\int_0^\infty \frac{x \sin mx}{x^2 + 1}\, dx = \frac{\pi}{2}\, e^{-m}.$$

   (c) The formula in (b) fails when $m = 0$. Why does this not contradict the basic Fourier transform theorem? (See "Remarks" on p. 143 of Constanda, or "Convergence theorems" in the section "More on Fourier transforms" of the class notes, or "Fourier's integral theorem" on p. 80 of Schaum's.)

8. *[Schaum's, p. 94, Ex. 5.30]* Use Parseval's identity and your knowledge of the Fourier sine and cosine transforms of e^{-x} $(x > 0)$ to evaluate

   (a) $\displaystyle\int_0^\infty \frac{dx}{(x^2 + 1)^2}$

   (b) $\displaystyle\int_0^\infty \frac{x^2\, dx}{(x^2 + 1)^2}$