Math. 401, Sec. 501 Spring, 2009

Homework 12, due April 24

1. *[Schaum's, p. 64, Ex. 3.22]* Let $\mathbf{r}$ be any three-dimensional vector, and $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ the standard unit vectors along the coordinate axes. Prove the following finite-dimensional analogues of the theorems about orthonormal bases of functions:

   (a) **(Parseval's equation)** $(\mathbf{r}\cdot\mathbf{i})^2 + (\mathbf{r}\cdot\mathbf{j})^2 + (\mathbf{r}\cdot\mathbf{k})^2 = \mathbf{r}\cdot\mathbf{r}$.

   (b) **(Bessel's inequality)** $(\mathbf{r}\cdot\mathbf{i})^2 + (\mathbf{r}\cdot\mathbf{j})^2 \le \mathbf{r}\cdot\mathbf{r}$.

   (c) **(least squares approximation)** $\mathbf{p} \equiv (\mathbf{r}\cdot\mathbf{i})\mathbf{i} + (\mathbf{r}\cdot\mathbf{j})\mathbf{j}$ is, of all vectors in the x–y plane, the closest to $\mathbf{r}$; furthermore, $\mathbf{p}$ is perpendicular to $\mathbf{r}-\mathbf{p}$.

2. *[Schaum's, p. 64, Ex. 3.23]* Suppose that one term in any orthonormal basis (i.e., a complete sequence of orthonormal functions) is omitted. (For example, leave $\sin(10x)$ out of the basis functions for the Fourier series on $(-\pi, \pi)$.) Call the resulting (amputated) sequence $\{\psi_n(x)\}$.

   (a) Can we expand an arbitrary function $f(x)$ as a series $\sum_n c_n\psi_n(x)$?

   (b) Is Parseval's identity always satisfied? Is it ever satisfied?

   (c) Is Bessel's inequality always satisfied?

   Justify your answers.

3. *[Logan, p. 195, Ex. 3.1(b)]* Find the eigenvalues and eigenfunctions for the problem

$$y'' + \lambda y = 0, \quad 0 < x < 1,$$

$$y(0) = 0, \qquad y(1) - y'(1) = 0.$$

4. *[Schaum's, p. 64, Ex. 3.43]* Show that the solution of the boundary value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad (0 < x < L,\ 0 < t),$$

$$\frac{\partial u}{\partial x}(t,0) = hu(t,0), \qquad \frac{\partial u}{\partial x}(t,L) = -hu(t,L), \qquad u(0,x) = f(x),$$

where h and L are constants, is

$$u(t,x) = \sum_{n=1}^{\infty} e^{-\omega_n{}^2 t}\, \frac{\omega_n\cos(\omega_n x) + h\sin(\omega_n x)}{(\omega_n{}^2 + h^2)L + 2h}\, c_n$$

where

$$c_n = \int_0^L f(x)(\omega_n\cos(\omega_n x) + h\sin(\omega_n x))\,dx$$

and the ω_n are solutions of

$$\tan(\omega L) = \frac{2h\omega}{\omega^2 - h^2}\,.$$

5. Using the δ-function method, construct a Green function to solve the problem

$$\frac{d^2y}{dx^2} + \omega^2 y = f(x), \qquad y'(0) = 0, \quad y'(2) = 0.$$

(Assume $\omega > 0$.) For what values of ω does no solution exist?

6. Using the δ-function method, construct a Green function to solve the problem

$$\frac{dy}{dt} - y = f(t), \qquad y(0) = 0.$$

Warning: In this case, G is *not* continuous at the location of the delta function.

7. Consider the Green function for the Laplace problem in the upper half plane,

$$G(x-z,y) = \frac{1}{\pi}\,\frac{y}{(x-z)^2+y^2}\,.$$

   (a) Verify that G satisfies Laplace's equation as a function of x and y.

   (b) Show that $\displaystyle\int_{-\infty}^{\infty} G(x-z,y)\,dz = 1$ for each fixed x and y.

   (c) Let $z = 1$ and sketch $G(x-1,y)$ as a function of x for three representative values of y.

   (d) Justify the claim that

$$\lim_{y\to 0^+} G(x-z,y) = \delta(x-z).$$

   This means that for any well-behaved function f (say f continuous and bounded),

$$\lim_{y\to 0^+} \int_{-\infty}^{\infty} G(x-z,y)\,f(z)\,dz = f(x).$$

   Hints: Write the integral as $\int_{-\infty}^{x-\delta} + \int_{x-\delta}^{x+\delta} + \int_{x+\delta}^{\infty}$, where δ is an arbitrarily small positive number. Show that the two outside integrals approach zero in the limit, using the assumption that f is bounded. In the middle integral, write $f(z) = f(x) + [f(z) - f(x)]$ and show that the integral involving the bracketed term can be assumed arbitrarily small, using the assumption that f is continuous.

   (e) Argue from (a) and (d) that G is the correct Green function for the problem — without appeal to Fourier transforms or any other external information. (That is, show that

$$u(x,y) \equiv \int_{-\infty}^{\infty} G(x-z,y)\,f(z)\,dz$$

   is the bounded solution of Laplace's equation in the upper half plane with the boundary data $u(x,0) = f(x)$.)