

**Homework 5, due February 25**

The first few problems here are a continuation of Problem 2 of the previous assignment, which concerned the equation

$$\frac{d^2 y}{dt^2} + [\omega^2 - V(t)]y = 0. \quad (1)$$

1. (a) In the equation

$$\frac{d^2 y}{dt^2} + \omega^2 V(t)y = 0 \quad (2)$$

make the change of time variable

$$\tau \equiv \omega \int_0^t \sqrt{V(\tilde{t})} d\tilde{t}.$$

*Hint:* As an intermediate step you should find that

$$\frac{d^2}{dt^2} = \omega^2 V(t) \frac{d^2}{d\tau^2} + \omega \frac{V'(t)}{2\sqrt{V(t)}} \frac{d}{d\tau}.$$

Show that the result is the equation of a harmonic oscillator subjected to small, slowly varying damping. (Define  $\epsilon = 1/\omega$ .)

- (b) Apply the two-variable method to the damped oscillator equation you found in (a). (For help, see J. D. Cole, *Perturbation Methods in Applied Mathematics*, Sec. 3.6.) You should get the same WKB approximation

$$V(t)^{-\frac{1}{4}} e^{\pm i\omega \int_0^t \sqrt{V(\tilde{t})} d\tilde{t}} \quad (***)$$

that was obtained in class by applying the two-variable method directly to (2).

2. (a) In (2) make the change of variables

$$z \equiv V^{\frac{1}{4}} y, \quad s \equiv \int_0^t \sqrt{V(\tilde{t})} d\tilde{t}.$$

[*Hint:* First carry out the change of dependent variable; then change the time variable in analogy with the previous problem.] Show that the result is a special case of (1).

- (b) Problem 2(a) shows that equations (1) and (2) are, at root, equivalent. Therefore, the solution to Problem 2(b) of the previous assignment should somehow match the first-order WKB solution (derived in notes and in Problem 1). Check this, by renaming  $V$  in (2) as  $1 - \epsilon^2 V$  and comparing the various solutions we've found.

3. The formula for the WKB approximation suggests that a solution which starts out proportional to  $\exp(+i\omega \int \sqrt{V})$  remains so for all  $t$  (rather than acquiring a component proportional to  $\exp(-i\omega \int \sqrt{V})$ ). This is actually true (through arbitrarily high finite order in  $1/\omega$ ) if  $V$  is a smooth function. However, a discontinuity in  $V$  or its derivative can cause “mixing” of positive- and negative-frequency complex exponentials. To see this: For each of the following functions  $V$ , write down the WKB solution that behaves as  $\exp(+i\omega \int \sqrt{V})$  on  $-\infty < t < 0$ . Match it to a WKB solution (linear combination of the two solutions (\*\*\*)) on  $0 < t < \infty$  by requiring that the solution and its derivative be continuous at 0, at least to lowest order.

$$(a) \quad V(t) = \begin{cases} 1 & \text{for } t < 0, \\ 2 & \text{for } t > 0. \end{cases}$$

$$(b) \quad V(t) = \begin{cases} 1 & \text{for } t < 0, \\ 1+t & \text{for } t > 0. \end{cases}$$

Comment on the difference between the two cases. (Relate the asymptotic dependence on  $\omega$  to the degree of smoothness (continuity and differentiability) of  $V$ .)

4. Consider the (second-order, linear) partial differential equation

$$i\epsilon \frac{\partial u}{\partial t} = -\frac{\epsilon^2}{2m} \nabla^2 u + V(\mathbf{x})u. \quad (\text{S})$$

Here  $\mathbf{x}$  is a variable in  $\mathbf{R}^3$  and  $\nabla$  is the gradient with respect to  $\mathbf{x}$ ;  $u(t, \mathbf{x})$  thus is a complex-valued function of four real variables. Make the ansatz

$$u(t, \mathbf{x}) = e^{iS(t, \mathbf{x})/\epsilon} [A_0(t, \mathbf{x}) + \epsilon A_1(t, \mathbf{x}) + \dots]$$

(where  $S$  and  $A_n$  are independent of  $\epsilon$ ). Show that the first two equations in the perturbation hierarchy are

$$\frac{\partial S}{\partial t} + \frac{1}{2m} (\nabla S)^2 + V = 0 \quad (\text{HJ})$$

and

$$\frac{\partial A_0}{\partial t} + \frac{1}{m} \nabla A_0 \cdot \nabla S + \frac{1}{2m} A_0 \nabla^2 S = 0. \quad (\text{T})$$

*Remarks:* In quantum mechanics (S) is the *Schrödinger equation* describing the behavior of a particle of mass  $m$ ;  $\epsilon$  is *Planck’s constant*, usually written as  $\hbar$ . Then (HJ) (a first-order, nonlinear equation for  $S$ ) is called the *Hamilton–Jacobi equation*, and in advanced classical mechanics courses it is shown that solving it is equivalent to solving the classical (nonquantum) equation of motion of the particle. Finally, (T) (a first-order, linear equation for  $A_0$ ) is called the *transport equation* because it can be solved by integration along the possible classical trajectories of the particle.