Math. 401, Sec. 501 Spring, 2009

Homework 8, due March 25

1. Suppose that the boundary conditions ("BC:") on p. 68 of the notes are replaced by

$$\frac{\partial w}{\partial x}(t,0) = 0, \qquad \frac{\partial w}{\partial x}(t,1) = 0.$$

What changes would be necessary in pp. 68–73 (and corresponding lecture on separation of variables in the heat equation)? *Warning:* Pay close attention to the possibility that λ may not always be positive.

2. *[Logan, p. 195, Ex. 3.4]* Find the Fourier sine series of $f(x) = 1$ on $0 \le x \le \pi$. To what value does the series converge at $x = 0$? at $x = \frac{\pi}{2}$?

3. *[Logan, p. 195, Ex. 3.9]* Solve by separation of variables (getting a Fourier sine series):

$$u_{xx} + u_{yy} = 0 \qquad \text{on} \quad 0 < x < a, \quad 0 < y < b,$$
$$u(x,0) = 0 \quad \text{and} \quad u(x,b) = f(x) \qquad \text{for } 0 < x < a,$$
$$u(0,y) = u(a,y) = 0 \qquad \text{for } 0 < y < b.$$

4. *[Schaum's, p. 46, Ex. 2.43]*

   (a) Show that for $-\pi < x < \pi$,

$$x = 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right).$$

   (b) By integrating the result of (a), show that for $-\pi \le x \le \pi$,

$$x^2 = \frac{\pi^2}{3} - 4\left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right).$$

   (Determine the constant of integration by requiring that the series formula give the right answer for $\int_0^\pi x^2\,dx$.)

   (c) By integrating once more and being careful about constants of integration, show that

$$x(\pi - x)(\pi + x) = 12\left(\frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \cdots\right).$$

5. *[Schaum's, p. 46, Ex. 2.37]*

   (a) Expand $f(x) = \cos x$, $0 < x < \pi$, in a Fourier sine series.

   (b) How should $f(x)$ be defined at $x = 0$ and $x = \pi$ so that the series will converge to $f(x)$ for $0 \leq x \leq \pi$?

6. *[Schaum's, p. 46, Ex. 2.50(a)]* Solve the boundary-value problem

$$\frac{\partial u}{\partial t} = 2\,\frac{\partial^2 u}{\partial x^2}\,, \qquad u(0,t) = u(4,t) = 0, \quad u(x,0) = 25x.$$

   It is understood that $0 < x < 4$ and $t > 0$.

Instructions for Exercises 7 and 8: J. B. Fourier was ridiculed by some of the mathematicians of his day when he first announced his discovery that an arbitrary function on the interval $0 < x < \pi$, such as $f(x) = x^2$, can be expanded in a series of sine functions. Some of the criticisms were like the two statements which follow. In each case explain in a short essay how the mathematicians were confused (and Fourier was right).

7. "x^2 is an even function; but any fool can see that a sum of sines will always be an odd function."

8. "x^2 is not zero at the right endpoint (π); but any fool can see that a sum of the functions $\sin nx$ will always vanish at $x = \pi$. The same criticism applies if we consider the limits of functions as $x \to \pi$ instead of the values of the functions when $x = \pi$."