Math. 401, Sec. 501 Spring, 2009

Homework 9, due April 1

1. *[Logan, p. 195, Ex. 3.2–3]*

   (a) Let $f(x) = x^2$ on $-\pi \le x \le \pi$ and $f(x + 2\pi) = f(x)$ for all x. Show that

   $$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n \frac{\cos(nx)}{n^2}.$$

   (b) Use this series to prove that $\dfrac{\pi^2}{12} = 1 - \dfrac{1}{4} + \dfrac{1}{9} - \dfrac{1}{16} + \cdots$.

2. *[Logan, p. 195, Ex. 3.6]* Find the ("full", exponential) Fourier series for the periodic function defined by

   $$f(x) = \begin{cases} 0 & \text{for } -\pi < x < 0, \\ 1 & \text{for } 0 \le x < \pi. \end{cases}$$

   Compare with the Fourier sine series you found for this function last week (Exercise 2 of Homework 8.) Explain the differences (if any) between them.

3. *[Schaum's, p. 46, Ex. 2.34–5(b)]* If

   $$f(x) = \begin{cases} -x & \text{when } -4 \le x \le 0, \\ x & \text{when } 0 \le x \le 4, \end{cases}$$

   and f is periodic with period 8, graph the function and find its Fourier series (using properties of even or odd functions whenever applicable). Also, tell where the discontinuities of f are located and to what value the series converges at each discontinuity.

4. *[Schaum's, p. 46, Ex. 2.34–5(c)]* If $f(x) = 4x$ for $0 < x < 10$ and f is periodic with period 10 (note: **not** 20), graph the function and find its Fourier series. Also, tell where the discontinuities of f are located and to what value the series converges at each discontinuity.

5. *[Schaum's, p. 46, Ex. 2.38]*

   (a) Expand $f(x) = \cos x$, $0 < x < \pi$, in a ("full") Fourier series.

   (b) Expand $f(x) = \cos x$, $0 < x < \pi$, in a Fourier cosine series.

   (c) Compare these results with the Fourier sine series you found for this function last week (Exercise 5 of Homework 8). Explain the differences (if any) among them.

6. *[Schaum's, p. 46, Ex. 2.40]* Show that for $0 \le x \le \pi$,

(a) $x(\pi - x) = \dfrac{\pi^2}{6} - \left(\dfrac{\cos 2x}{1^2} + \dfrac{\cos 4x}{2^2} + \dfrac{\cos 6x}{3^2} + \cdots\right).$

(b) $x(\pi - x) = \dfrac{8}{\pi}\left(\dfrac{\sin x}{1^3} + \dfrac{\sin 3x}{3^3} + \dfrac{\sin 5x}{5^3} + \cdots\right).$

7. *[Schaum's, p. 46, Ex. 2.41)]* Use the results of Exercise 6 to show (by judicious choices of x) that

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}\,.$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}\,.$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^3} = \frac{\pi^3}{32}\,.$

8. *[Schaum's, p. 46, Ex. 2.46 and 2.48)]* Use Parseval's identity along with Exercise 4 of Homework 8 and Exercise 6 of this assignment to show (in any convenient order)

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}\,.$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}\,.$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{\pi^4}{96}\,.$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} = \frac{\pi^6}{960}\,.$