Math. 401 (Fulling) 27 February 2009

Test A – Solutions

"Up **to** [a certain order]" *means* "stop right before calculating the term of that order."
"Up **through** [a certain order]" *means* "**do** calculate a term of that order."

Calculators may be used for simple arithmetic operations only!

1. *(32 pts.)* Find approximations to all the roots of

$$\epsilon x^3 + (1+2\epsilon)x^2 + \epsilon x - 9 = 0 \qquad (\epsilon \to 0)$$

up through order ϵ^1.

First try regular perturbation theory: $x \sim x_0 + \epsilon x_1$.

$$\begin{aligned} 0 &= \epsilon({x_0}^3 + \cdots) + (1+2\epsilon)({x_0}^2 + 2\epsilon x_0 x_1 + \cdots) + \epsilon(x_0 + \cdots) - 9 \\ &= {x_0}^2 - 9 + \epsilon({x_0}^3 + 2{x_0}^2 + 2\epsilon x_0 x_1 + x_0) + O(\epsilon^2)\,. \end{aligned}$$

$O(\epsilon^0)$: ${x_0}^2 = 9 \Rightarrow x_0 = \pm 3$.

$O(\epsilon^1)$: First let's cancel an x_0: ${x_0}^2 + 2x_0 + 1 = -2x_1$. Thus

$$x_1 = -\tfrac{1}{2}(9 \pm 6 + 1) = -5 \mp 3\,.$$

So we have found two roots,

$$x \sim 3 - 8\epsilon \quad \text{and} \quad x \sim -3 - 2\epsilon\,.$$

To get the remaining root, try singular perturbation theory. Balance the x^3 term against the first term without an ϵ: $\epsilon x^3 \sim x^2 \Rightarrow x \sim \epsilon^{-1}$. Therefore, let

$$x = \frac{\overline{x}}{\epsilon}\,.$$

The equation becomes

$$\frac{\overline{x}^3}{\epsilon^2} + \frac{(1+2\epsilon)\overline{x}^2}{\epsilon^2} + \overline{x} - 9 = 0\,,$$

or

$$\overline{x}^3 + \overline{x}^2 + 2\epsilon\overline{x}^2 + \epsilon^2\overline{x} - 9\epsilon^2 = 0\,.$$

Now $\overline{x} \sim \overline{x}_0 + \epsilon\overline{x}_1$ implies

$$0 = {\overline{x}_0}^3 + 3\epsilon{\overline{x}_0}^2\overline{x}_1 + {\overline{x}_0}^2 + 2\epsilon\overline{x}_0\overline{x}_1 + 2\epsilon{\overline{x}_0}^2 + O(\epsilon^2)\,.$$

$O(\epsilon^0)$: ${\overline{x}_0}^3 + {\overline{x}_0}^2 = 0 \Rightarrow \overline{x}_0 = -1$.

$O(\epsilon^1)$: $0 = 3{\overline{x}_0}^2\overline{x}_1 + 2\overline{x}_0\overline{x}_1 + 2{\overline{x}_0}^2 = 3\overline{x}_1 - 2\overline{x}_1 + 2 \Rightarrow \overline{x}_1 = -2$. Thus

$$\overline{x} \sim -1 - 2\epsilon \Rightarrow x \sim -\frac{1}{\epsilon} - 2\,.$$

Check: If you plug the three roots into the equation, you will find that the two-term approximation to the singular root satisfies the equation up to terms of order ϵ^0 (that is, terms of orders ϵ^{-2} and ϵ^{-1} cancel), and the two regular cases satisfy the equation up to terms of order ϵ^2. This is exactly the degree of accuracy we should expect.

2. *(15 pts.)* Pronounce each of the following assertions true or false, and write something to explain your judgment.

(a) $$\frac{\epsilon^2}{1-\cos\epsilon} = O(\epsilon) \quad \text{as } \epsilon \to 0.$$

False. For small ϵ the denominator behaves like $\frac{1}{2}\epsilon^2$, so the ratio on the left does not approach 0.

(b) $$\ln x = o(x) \quad \text{as } x \to +\infty.$$

True. Look at $\dfrac{\ln x}{x}$. By l'Hôpital's rule, it approaches 0, as required by the definition of o.

(c) $$\frac{\sin\epsilon}{\epsilon} = O(\epsilon^0) \text{ [i.e., is bounded]} \quad \text{as } \epsilon \to 0.$$

True. The left-hand side approaches 1, so it is bounded near $\epsilon = 0$.

3. *(40 pts.)* Find a two-term solution (up through order ϵ) by the distorted-time method:

$$\frac{d^2y}{dt^2} + 4y + \epsilon y^3 = 0, \qquad y(0) = 1, \quad \frac{dy}{dt}(0) = 0 \qquad (\epsilon \to 0).$$

You may find some of the formulas on the back side helpful.

Let $y \sim y_0 + \epsilon y_1$, and

$$\tau \sim t + \epsilon\omega_1 t \;\Rightarrow\; \frac{d}{dt} = \bigl(1 + \epsilon\omega_1 + O(\epsilon^2)\bigr)\frac{d}{d\tau}\,.$$

Make the time substitution first:

$$\begin{aligned} 0 &\sim (1+\epsilon\omega_1)^2 \frac{d^2y}{d\tau^2} + 4y + \epsilon y^3 \\ &\sim \frac{d^2y}{d\tau^2} + 2\epsilon\omega_1 \frac{d^2y}{d\tau^2} + 4y + \epsilon y^3\,. \end{aligned}$$

Now the y substitution:

$$\begin{aligned} 0 &\sim \frac{d^2y_0}{d\tau^2} + \epsilon\frac{d^2y_1}{d\tau^2} + 2\epsilon\omega_1 \frac{d^2y_0}{d\tau^2} + 4y_0 + 4\epsilon y_1 + \epsilon {y_0}^3 \\ &= \frac{d^2y_0}{d\tau^2} + 4y_0 + \epsilon\left[\frac{d^2y_1}{d\tau^2} + 4y_1 + 2\omega_1\frac{d^2y_0}{d\tau^2} + {y_0}^3\right]. \end{aligned}$$

And, finally, transform the initial conditions:

$$1 \sim y_0(0) + \epsilon y_1(0)\,,$$

$$0 \sim (1+\epsilon\omega_1)\left[\frac{dy_0}{d\tau} + \epsilon\,\frac{dy_1}{d\tau}\right] \sim \frac{dy_0}{d\tau}(0) + \epsilon\left[\frac{dy_1}{d\tau}(0) + \omega_1\frac{dy_0}{d\tau}(0)\right].$$

$O(\epsilon^0)$: $\quad \dfrac{d^2y_0}{d\tau^2} + 4y_0 = 0\,, \quad y_0(0) = 1\,, \quad \dfrac{dy_0}{d\tau} = 0\,.$

Thus $y_0(\tau) = \cos(2\tau)$.

$O(\epsilon^1)$: $$\frac{d^2y_1}{d\tau^2} + 4y_1 = -2\omega_1 \frac{d^2y_0}{d\tau^2} - {y_0}^3$$

$$= 8\omega_1 \cos(2\tau) - \cos^3(2\tau)$$
$$= 8\omega_1 \cos(2\tau) - \tfrac{1}{4}\cos(6\tau) - \tfrac{3}{4}\cos(2\tau)\,.$$

The $\cos(2\tau)$ term is resonant, so to remove it we require $8\omega_1 - \frac{3}{4} = 0$, or $\omega_1 = 3/32$. The initial conditions in this order are

$$y_1(0) = 0\,, \quad \frac{dy_2}{d\tau}(0) = -\omega_1 \frac{dy_0}{d\tau}(0) = 0\,.$$

The nonresonant forcing term remains, so we will have a nontrivial $y_{1\mathrm{p}}$, of the form

$$y_1(\tau) = A\cos(6\tau) + B\sin(6\tau) \;\Rightarrow\; \frac{d^2y_1}{d\tau^2} = -36A\cos(6\tau) - 36B\sin(6\tau)\,.$$

Thus

$$\frac{d^2y_1}{d\tau^2} + 4y_1 = -32A\cos(6\tau) - 32B\sin(6\tau)$$

must be required to equal $-\frac{1}{4}\cos(6\tau)$, so

$$A = \frac{1}{128}\,, \quad B = 0\,.$$

To get the complete y_1, add a solution of the homogeneous equation:

$$y_1(\tau) = \frac{1}{128}\cos(6\tau) + c_1\cos(2\tau) + c_2\sin(2\tau)\,.$$

The initial conditions then require

$$c_1 = -\frac{1}{128}\,, \quad c_2 = 0\,.$$

Thus, finally,

$$y \sim \cos(2\tau) + \frac{\epsilon}{128}[\cos(6\tau) - \cos(2\tau)] \quad \text{where} \quad \tau \sim \left(1 + \frac{3\epsilon}{32}\right) t\,.$$

4. *(13 pts.)* Describe in words (and a few symbols) how you would attack these equations. (For full credit you must do more than just give the method a name, but you won't have time to carry out extensive calculations.)

["Doing more" constitutes an "essay question" and hence is not included in the answer key.]

(a) $$\frac{d^2y}{dt^2} + \epsilon^{-2}e^t\, y = 0$$

Use one of the variants of the WKB or Liouville–Green approximation.

(b) $$\frac{d^2y}{dt^2} + 4y + \epsilon\left(\frac{dy}{dt}\right)^3 = 0$$

Make a two-time ansatz (because of the damping caused by the first-derivative term).

$$\sin x \sim x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots, \qquad \cos x \sim 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

$$(1+x)^p \sim 1 + px + \frac{p(p-1)}{2}\,x^2 + \frac{p(p-1)(p-2)}{3!} + \cdots$$

$$\cos^2 x = \tfrac{1}{2} + \tfrac{1}{2}\cos(2x), \qquad \sin^2 x = \tfrac{1}{2} - \tfrac{1}{2}\cos(2x)$$

$$\sin x \cos y = \tfrac{1}{2}\sin(x-y) + \tfrac{1}{2}\sin(x+y)$$

$$\cos x \cos y = \tfrac{1}{2}\cos(x-y) + \tfrac{1}{2}\cos(x+y)$$

$$\sin x \sin y = \tfrac{1}{2}\cos(x-y) - \tfrac{1}{2}\cos(x+y)$$

$$\cos^3 x = \tfrac{1}{4}\cos(3x) + \tfrac{3}{4}\cos x, \qquad \sin^3 x = -\tfrac{1}{4}\sin(3x) + \tfrac{3}{4}\sin x$$

$$\cos^2 x \sin x = \tfrac{1}{4}\sin(3x) + \tfrac{1}{4}\sin x, \qquad \cos x \sin^2 x = -\tfrac{1}{4}\cos(3x) + \tfrac{1}{4}\cos x$$