Math. 401 (Fulling) 13 April 2009

Test B – Solutions

Calculators may be used for simple arithmetic operations only!

1. *(25 pts.)* Let $g(x) = \begin{cases} 1 & \text{for } 0 < x < \frac{1}{2}\,, \\ 0 & \text{for } \frac{1}{2} < x < 1\,. \end{cases}$

You should be able to answer (b) and (c) even if you have trouble with the details of (a).

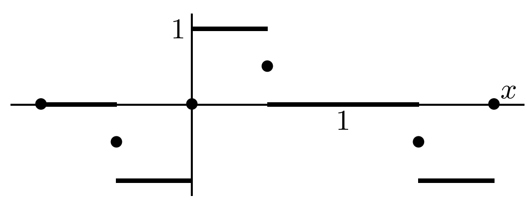
(a) Find the Fourier *sine* series of g (on interval $(0, 1)$).
(Simplify the coefficient formulas as much as you can.)

In the standard notation, $L = 1$.

$$g(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x),$$

$$\begin{aligned}
b_n &= \frac{2}{1}\int_0^1 \sin(n\pi x) g(x)\,dx = 2\int_0^{1/2} \sin(n\pi x)\,dx \\
&= -\left.\frac{2\cos(n\pi x)}{n\pi}\right|_0^{1/2} = -\frac{2}{n\pi}\left[\cos\left(\frac{n\pi}{2}\right) - \cos 0\right] \\
&= \begin{cases} \dfrac{2}{n\pi} & \text{if } n \text{ is odd,} \\ \dfrac{2}{n\pi}\bigl(1 - (-1)^m\bigr) & \text{if } n = 2m, \end{cases} \\
&= \begin{cases} \dfrac{2}{n\pi} & \text{if } n \text{ is odd,} \\ \dfrac{4}{n\pi} & \text{if } n = 2, 6, 10, \ldots \text{ (i.e., } n = 2 \bmod 4), \\ 0 & \text{if } n = 4, 8, \ldots \text{ (i.e., } n = 0 \bmod 4). \end{cases}
\end{aligned}$$

(b) Sketch the graph of the function to which the series converges for $-1 \le x \le 2$.



(c) Does the series converge when $x = \frac{1}{2}$? To what? By what type of convergence?

Yes, the series converges pointwise (not uniformly) to the value $\frac{1}{2}$.

2. *(30 pts.)* Solve the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad (0 < t < \infty\,,\ 0 < x < 1)$$

with boundary conditions

$$u(t,0) = 0\,, \qquad u(t,1) = 2$$

and initial data

$$u(0,x) = f(x) \equiv \begin{cases} 1 + 2x & \text{for } 0 < x < \frac{1}{2}\,, \\ 2x & \text{for } \frac{1}{2} < x < 1\,. \end{cases}$$

You may skip some routine steps if you clearly explain your conclusions.

Because we have a nonhomogeneous boundary condition, we look first for a steady-state solution satisfying

$$0 = \frac{d^2 v}{dx^2}\,, \qquad v(0) = 0\,, \qquad v(1) = 2\,.$$

We find $v(x) = Ax + B$ and hence $B = 0$, $A = 2$, so

$$v(x) = 2x\,.$$

Now

$$u(t,x) = v(x) + w(t,x)$$

where w satisfies

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2}\,, \qquad w(t,0) = 0\,, \qquad w(t,1) = 0\,,$$

$$w(0,x) = g(x) \equiv f(x) - v(x) = \begin{cases} 1 & \text{for } 0 < x < \frac{1}{2}\,, \\ 0 & \text{for } \frac{1}{2} < x < 1\,. \end{cases}$$

By a fantastic coincidence this is the same g as in the previous problem.

We expect the solution to be a series of separated solutions of the form $X_n(x)T_n(t)$, where

$$\frac{T'}{T} = -{\omega_n}^2 = \frac{X''}{X} \qquad \text{and} \qquad X(0) = 0 = X(1).$$

From previous work we know that

$$\omega_n = \frac{n\pi}{1}\,, \qquad X_n(x) = \sin(n\pi x) \qquad T_n(t) = e^{-n^2\pi^2 t}\,.$$

So

$$w(t,x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) e^{-n^2\pi^2 t}\,.$$

To determine the b_n we set $t = 0$:

$$g(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x).$$

And now we can just read off the formulas for b_n from the previous solution.

3. *(Essay – 10 pts.)* How would you modify Qu. 2 (in small ways) to get a problem whose solution involves a Fourier *cosine* series? Discuss the physical interpretation of the difference(s) between the original and new problems.

The cosine series corresponds to Neumann boundary conditions — i.e., specify the derivatives of u at the endpoints instead of the values of u itself. However, in that case there is a complication: no steady-state condition exists unless the two boundary values are equal, say

$$\frac{\partial u}{\partial x}(t,0) = F = \frac{\partial u}{\partial x}(t,1)\,.$$

[In that case the steady-state solution turns out to be nonunique, because the $n=0$ (constant) mode of the homogeneous problem is also time-independent (so there is no unambiguous way to separate it from the steady-state solution). The final solution of the full problem is unique, because changing v forces a corresponding change in a_0 for given f. (All the stuff inside these square brackets is extra credit. If you set $F=0$ in your answer, I'm satisfied.)]

The Dirichlet conditions in the original problem describe fixed endpoint temperatures. Neumann conditions describe fixed heat fluxes at the end points. In particular, $F=0$ means that the ends are thermally insulated. If the two fluxes are unequal, heat must build up (or be depleted) inside the bar, which explains why there is no steady-state solution in that case.

4. *(35 pts.)* Find the lowest-order outer, inner, and composite (also called "uniform") solutions for

$$\epsilon\,\frac{d^2y}{dt^2} - \frac{dy}{dt} + \frac{1}{2}\,y^3 = 0\,, \quad 0 < t < \frac{1}{2}\,, \quad 0 < \epsilon \ll 1\,,$$

with the boundary conditions $y(0) = 1\,,\ y\left(\frac{1}{2}\right) = 4\,.$

Since the first- and second-derivative terms have opposite signs, we expect to have a boundary layer on the right, near $t = \frac{1}{2}$.

The outer solution is given by regular perturbation theory with the boundary condition at $t=0$:

$$\frac{dy}{dt} = \frac{1}{2}\,y^3 \ \Rightarrow\ 2\int \frac{dy}{y^3} = \int dt$$
$$\Rightarrow\ -\frac{1}{y^2} = t - C \ \Rightarrow\ y^2 = \frac{1}{C-t}\,,$$

where the negative sign on the constant was inserted to make the next steps neat:

$$y_{\text{o}}(t) = \frac{1}{\sqrt{C-t}}\,,$$

and $1 = y_{\text{o}}(0) = C^{-1/2} \ \Rightarrow\ C = 1$. Thus

$$y_{\text{o}}(t) = \frac{1}{\sqrt{1-t}}\,,$$

To find the inner solution we rescale in the neighborhood of $t = \frac{1}{2}$. Define

$$\tau = \frac{\frac{1}{2} - t}{\epsilon} > 0\,, \qquad \text{so that} \quad \frac{d}{dt} = -\frac{1}{\epsilon}\,\frac{d}{d\tau}\,.$$

Then the equation becomes (after multiplication by ϵ)

$$\frac{d^2y}{d\tau^2} + \frac{dy}{d\tau} + \frac{\epsilon}{2}\, y^3 = 0\,.$$

To lowest order, we get a first-order differential equation for $dy/d\tau$:

$$\frac{d^2y}{d\tau^2} + \frac{dy}{d\tau} = 0 \;\Rightarrow\; \frac{dy}{d\tau} = C_1 e^{-\tau}\,.$$

Therefore,

$$y_{\rm i} = -C_1 e^{-\tau} + C_2\,.$$

Now by construction $\tau = 0$ when $t = \frac{1}{2}$, so the other boundary condition is $4 = -C_1 + C_2$, or $C_2 = C_1 + 4$. Thus

$$y_{\rm i} = -C_1 e^{-\tau} + C_1 + 4 = -C_1 \exp\left[\frac{1}{\epsilon}\left(t - \frac{1}{2}\right)\right] + C_1 + 4\,.$$

To determine the last constant we must match the two solutions in an intermediate limit. Define

$$\eta = \sqrt{\epsilon}\,\tau = \frac{1}{\sqrt{\epsilon}}\left(\frac{1}{2} - t\right) > 0\,,$$

so that

$$\tau = \frac{1}{\sqrt{\epsilon}}\,\eta\,, \qquad t = \frac{1}{2} - \sqrt{\epsilon}\,\eta\,.$$

Take the limit $\epsilon \to 0$ with η fixed:

$$y_{\rm o} = \frac{1}{\sqrt{1 - \frac{1}{2} + \sqrt{\epsilon}}} \to \sqrt{2}\,,$$

$$y_{\rm i} = -C_1 e^{-\eta/\sqrt{\epsilon}} + C_1 + 4 \to C_1 + 4\,.$$

These must agree, so $C_1 = \sqrt{2} - 4$. Thus

$$y_{\rm i}(t) = (4 - \sqrt{2}) \exp\left[\frac{1}{\epsilon}\left(t - \frac{1}{2}\right)\right] + \sqrt{2}\,.$$

Finally, form the composite solution as

$$\begin{aligned} y(t) &= y_{\rm o} + y_{\rm i} - \sqrt{2} \\ &= \frac{1}{\sqrt{1-t}} + (4 - \sqrt{2}) \exp\left[\frac{1}{\epsilon}\left(t - \frac{1}{2}\right)\right]\,. \end{aligned}$$