Math. 412 (Fulling, 2004), **Homework 8**

1. Haberman 9.3.22. (Note: For first-order equations, G is *not* continuous at the location of the delta function. Why not?) Use the result to solve $y' + y = f(t)$, $y(0) = 0$, for arbitrary f.
2. Solve the wave equation in $\mathbf{R}^2$ by Fourier transforms. (Recycle and extend your solution to Haberman 10.4.10, with $c = 1$ but $\frac{\partial u}{\partial t}(x, 0) = g(x) \neq 0$.) Rearrange the result into the form

$$u(x, t) = \int_{-\infty}^{\infty} dy \, [W(x - y, t) f(y) + V(x - y, t) g(y)] \, ,$$

   where f and g are the initial data and W and V are certain integral expressions. What identities for the Fourier transforms of delta functions and step functions do you need to assume in order to get your answer to agree with the one on p. 80 of the notes? Try to boil it down to simple formulas for functions (or distributions) of *one* variable, not three.
3. Consider the Green function for the Laplace problem in the upper half plane,

$$G(x - z, y) = \frac{1}{\pi} \frac{y}{(x - z)^2 + y^2} \, .$$

   (a) Verify that G satisfies Laplace's equation as a function of x and y.

   (b) Show that $\displaystyle\int_{-\infty}^{\infty} G(x - z, y) \, dz = 1$ for each fixed x and y.

   (c) Let $z = 1$ and sketch $G(x - 1, y)$ as a function of x for three representative values of y.

   (d) Justify the claim that

$$\lim_{y \to 0^+} G(x - z, y) = \delta(x - z).$$

   This means that for any well-behaved function f (say f continuous and bounded),

$$\lim_{y \to 0^+} \int_{-\infty}^{\infty} G(x - z, y) \, f(z) \, dz = f(x).$$

   Hints: Write the integral as $\int_{-\infty}^{x-\delta} + \int_{x-\delta}^{x+\delta} + \int_{x+\delta}^{\infty}$, where δ is an arbitrarily small positive number. Show that the two outside integrals approach zero in the limit, using the assumption that f is bounded. In the middle integral, write $f(z) = f(x) + [f(z) - f(x)]$ and show that the integral involving the bracketed term can be assumed arbitrarily small, using the assumption that f is continuous.

   (e) Argue (a bit loosely, perhaps) from (a) and (d) that G is the correct Green function for the problem — without appeal to Fourier transforms or any other external information. (That is, show that

$$u(x, y) \equiv \int_{-\infty}^{\infty} G(x - z, y) \, f(z) \, dz$$

   is the bounded solution of Laplace's equation in the upper half plane with the boundary data $u(x, 0) = f(x)$.)