Math. 412 (Fulling) 24 September 2004

# Test A – Solutions

**Calculators may be used for simple arithmetic operations only!**

1. *(40 pts.)* Let $f(x) = x$ for $0 \le x < 1$.

(a) Find the Fourier sine series for $f$ (with $[0,1]$ as the basic interval).

Take $L = 1$ in the usual formulas:

$$x \sim \sum_{n=1}^{\infty} b_n \sin(n\pi x),$$

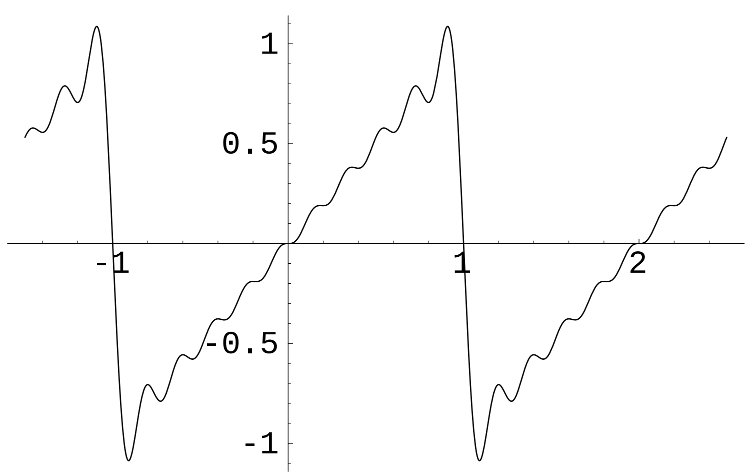
$$\begin{aligned} b_n &= 2\int_0^1 x\sin(n\pi x)\,dx \\ &= -\frac{2}{n\pi}\left[ x\cos(n\pi x)\big|_0^1 - \int_0^1 \cos(n\pi x)\,dx \right] \\ &= -\frac{2}{n\pi}\left[ \cos(n\pi) - \frac{1}{n\pi}\sin(n\pi x)\big|_0^1 \right] \\ &= \frac{(-1)^{n+1}2}{n\pi}. \end{aligned}$$

(b) Over the interval $[-1.5, 2.5]$, sketch the function to which the series converges.

[See Haberman, Fig. 3.3.9.]

(c) Sketch what you imagine the 10th partial sum of the series looks like. (Don't try to actually calculate it.)

Here is an exact plot:




(d) Does the series converge

(i) uniformly?

NO. The extended function (see (b)) is not continuous (at the odd integers).

(ii) pointwise?

YES. The extended function is piecewise smooth. (At the odd integers, the value of the extended function must be defined to be $0$ .)

(iii) in the mean?

YES. The function is square-integrable (on the original interval, $[0,1]$ ).

(e) Briefly describe how the answers to (b), (c), and (d) would change if we studied the cosine series instead.

The extended function is now even instead of odd; its graph is Fig. 3.3.14 of Haberman. This function is continuous, so the convergence is uniform and there is no Gibbs overshoot in the partial sums.

2. *(30 pts.)* Solve the heat conduction problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad (0 < x < 1, \quad 0 < t < \infty),$$

$$u(0,t) = 0 = u(1,t) \qquad (0 < t < \infty),$$

$$u(x,0) = x \qquad (0 < x < 1).$$

Stop when you can say “And now continue as in Question 1, above.”

First search for separated solutions: $u = X(x)T(t)$ . Then

$$\frac{T'}{T} = \frac{X''}{X} = -k^2 .$$

The eigenvalue problem is

$$X'' + k^2 X = 0, \quad X(0) = 0 = X(1).$$

From previous experience we know that $k^2$ must be real and positive. Thus

$$X(x) = \sin(kx) \quad \text{and (since } X(1) = 0\text{ )} \quad k = n\pi \quad (\, n = \text{ positive integer}).$$

The equation for $T$ is now $T' = -k^2 T$ , with solution

$$T(t) = (\text{constant } \times\,)\; e^{-k^2 t}.$$

These elementary solutions satisfy the PDE and the boundary conditions but not the initial condition.

Second, the most general solution is a superposition of these:

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) e^{-n^2\pi^2 t}.$$

The initial condition becomes

$$x = \sum_{n=1}^{\infty} b_n \sin(n\pi x).$$

And now continue as in Question 1, above.

3. *(30 pts.)* Consider the wave propagation problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad (0 < x < \infty, \quad -\infty < t < \infty),$$

$$u(0,t) = 0 \qquad (-\infty < t < \infty),$$

$$u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = -10(x-1)e^{-5(x-1)^2} \qquad (0 < x < \infty).$$

Answer the following in whichever order you prefer:

(a) Find a formula for the solution $u(x,t)$.

We need d'Alembert's formula with $f(x) = u(x,0) = 0$ and $g(x) = \frac{\partial u}{\partial t}(x,0)$ as given. We can use either of the two standard versions of the formula, or recover it from first principles as follows: Let $u(x,t) = B(x-t) + C(x+t)$. Then

$$\begin{aligned} 0 &= B(x) + C(x), \\ g(x) &= -B'(x) + C'(x). \end{aligned}$$

Integrate the second equation:

$$-B(x) + C(x) = G(x) = \int g(x)\,dx.$$

Thus

$$-B(x) = C(x) = \tfrac{1}{2}G(x).$$

The solution so far is valid for initial data given on $(-\infty, \infty)$. To reduce the boundary-value problem to that case we need to take the odd extension of $g$, or the even extension of $G$, where

$$G(x) = e^{-5(x-1)^2} \quad \text{for } x > 0\,.$$

A formula valid for both signs of $x$ is

$$G(x) = e^{-5(|x|-1)^2}.$$

So the solution is

$$u(x,t) = \frac{1}{2}\left[e^{-5(|x+t|-1)^2} - e^{-5(|x-t|-1)^2}\right].$$

(b) Roughly sketch the solution as a function of $x$ for $t = \frac{1}{2}$ and for $t = 2$. (*Hint:* $\frac{\partial u}{\partial t}(x, 0)$ has been cunningly chosen so that its antiderivative is a sharply peaked function.)

In the physical region, initially, the right-moving pulse has opposite sign from the left-moving pulse, so that they will cancel at $t = 0$. At $t = 1$ the left-moving pulse bounces off the wall, and, as always for the Dirichlet condition, the reflected pulse is inverted. (A close look at the formula in (a) confirms that the right-moving pulse that starts in the unphysical region and eventually becomes the reflected pulse does indeed have the negative sign.)