Math. 412 12 December 2006

Final Examination – Solutions

Calculators may be used for simple arithmetic operations only!

Some possibly useful information

Laplacian operator in spherical coordinates ($\theta =$ polar angle, $\phi =$ azimuthal angle):

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}.$$

Spherical harmonics satisfy

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]Y_l^m(\theta,\phi) = -l(l+1)Y_l^m(\theta,\phi).$$

Legendre's equation:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + l(l+1)\Theta = 0 \quad \text{has a nice solution } P_l(\cos\theta)\,.$$

Bessel's equation:

$$\frac{\partial^2 Z}{\partial z^2} + \frac{1}{z}\frac{\partial Z}{\partial z} + \left(1 - \frac{n^2}{z^2}\right)Z = 0 \quad \text{has solutions } J_n(z) \text{ and } Y_n(z)\,.$$

Airy's equation:

$$\frac{\partial^2 y}{\partial z^2} - zy = 0 \quad \text{has a nice solution } \mathrm{Ai}(z)\,;$$

$\mathrm{Ai}(z)$ is bounded and oscillatory as $z \to -\infty$ and decreases very fast as $z \to +\infty$.

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1. *(40 pts.)* Solve Laplace's equation in the **exterior** of a sphere,

$$\nabla^2 u = 0 \quad \text{for} \quad R < r < \infty\,, \qquad u(R,\theta,\phi) = f(\theta)\sin(2\phi)\,.$$

Skip steps if you know the answer. (Note: Haberman would write (ρ,ϕ,θ) for (r,θ,ϕ) .)

Knowing that the spherical harmonics were invented precisely to solve this type of problem, we dispense with the (θ,ϕ) separation and look for solutions of the form

$$u(r,\theta,\phi) = R(r)Y_l^m(\theta,\phi)$$

Then the top two of the useful formulas yield

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{l(l+1)}{r^2}R = 0\,.$$

The independent solutions of this equation are r^l and $r^{-(l+1)}$, and it is the second of these that is well-behaved outside the sphere.

So the full solution is

$$u(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} c_m^l r^{-(l+1)} Y_l^m(\theta,\phi)\,.$$

Because Y_l^m has a factor $e^{im\phi}$, to reproduce the data function $\sin(2\phi)$ we will need only $m=\pm 2$, and hence only $l \ge 2$:

$$u(r,\theta,\phi) = \sum_{l=2}^{\infty} r^{-(l+1)}[c_2^l Y_l^2 + c_{-2}^l Y_l^{-2}]\,,$$

where

$$f(\theta)\sin(2\phi) = \sum_{l=2}^{\infty} R^{-(l+1)}[c_2^l Y_l^2 + c_{-2}^l Y_l^{-2}]\,.$$

The spherical harmonics are defined to be orthonormal, so

$$c_{\pm 2}^l = R^{+(l+1)} \int_0^\pi d\theta \int_0^{2\pi} \sin\theta\, d\phi\, Y_l^{\pm 2}(\theta,\phi)^* f(\theta)\sin(2\phi).$$

(As usual, the entire integrand is understood to be inside the integrals; the differentials are next to their respective integral signs only for clarity.)

By introducing the formula for Y_l^m in terms of $e^{im\phi}$ and an associated Legendre function $P_l^m(\cos\theta)$ we could evaluate the integral over ϕ explicitly. However, we would then need to look up the normalization factor for the Legendre function, so let's stop here.

2. *(40 pts.)* By the method of your choice, solve the wave equation on the half-line,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad \text{for} \quad 0 < x < \infty\,, \qquad u(0,t) = 0\,, \quad u(x,0) = 0\,, \quad \frac{\partial u}{\partial t}(x,0) = g(x)\,.$$

Fourier method: By separation of variables (or going directly to a Fourier sine transform) we arrive at

$$u(x,t) = \int_0^\infty \sin(kx)[a(k)\cos(kt) + b(k)\sin(kt)]\,dk\,.$$

Since $u(0,t)=0$, $a(k)=0$. We have

$$g(x) = \int_0^\infty b(k)\sin(kx)k\cos(kt)\,dk\,.$$

Therefore,

$$b(k) = \frac{2}{\pi k}\int_0^\infty \sin(kx)g(x)\,dx\,.$$

D'Alembert method: We must have

$$u(x,t) = B(x-t) + C(x+t)\,.$$

This gives

$$0 = f(x) = B(x) + C(x)\,,$$
$$g(x) = -B'(x) + C'(x)\,.$$

Write $G(x)$ for an arbitrary antiderivative of g: $G(x) = -B(x) + C(x)$. Then

$$u(x,t) = \tfrac{1}{2}[G(x+t) - G(x-t)] = \frac{1}{2}\int_{x-t}^{x+t} g(z)\,dz\,.$$

This makes sense if g is defined for all z, negative as well as positive. We can match the Dirichlet boundary condition by taking the odd extension of g (or the even extension of G): $g(-z) = -g(z)$. (In more detail, we must have $0 = B(-t) + C(t)$, which is consistent with formulas above if and only if G is even and g is odd.)

3. *(40 pts.)* Consider $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} - xu$ on $0 < x < \infty$ with the conditions $u(0,t) = 0$ and $u(x,0) = f(x)$:

   (a) Solve by separation of variables. (The spectrum of eigenvalues will be discrete. You can't solve for the eigenvalues but you can write down an equation that determines them.)

Try $u_{\text{sep}} = T(t)X(x)$, getting

$$\frac{T'}{T} = \frac{X''}{X} - x = -\lambda.$$

Then $T = e^{-\lambda t}$ and

$$\frac{d^2X}{dx^2} - (x-\lambda)X = 0.$$

This is Airy's equation with $z = x - \lambda$, so the eigenfunctions (not normalized) are

$$\phi_n(x) = \mathrm{Ai}(x - \lambda_n)\,.$$

(The other Airy function is not admissible because it increases very fast as $z \to +\infty$.) Finally, to satisfy the boundary condition we need

$$0 = \phi_n(0) = \mathrm{Ai}(-\lambda_n)\,.$$

There is a discrete, infinite list of solutions of this equation, since Ai is oscillatory at negative arguments.

Now the solution of the problem is

$$u(x,t) = \sum_{n=1}^{\infty} c_n \,\mathrm{Ai}(x-\lambda_n) e^{-\lambda_n t}\,.$$

The coefficient formula is

$$c_n = \frac{\int_0^\infty f(x)\,\mathrm{Ai}(x-\lambda_n)\,dx}{\int_0^\infty \mathrm{Ai}(x-\lambda_n)^2\,dx}\,.$$

The denominator can be simplified to $\int_{-\lambda_n}^{\infty} \mathrm{Ai}(z)^2\,dz$.

(b) Write the solution in Green-function form. (If you don't know the answer to (a), just introduce eigenfunctions $\phi_n(x)$ and eigenvalues λ_n, but explain how to normalize the eigenfunctions.)

Rewrite the coefficient formula with y in place of x as the variable of integration; plug into the solution formula and reverse the order of integration.

$$u(x,t) = \int_0^\infty G(x,y,t) f(y)\, dy$$

with

$$G(x,y,t) = \sum_{n=0}^\infty \frac{\mathrm{Ai}(x-\lambda_n)\,\mathrm{Ai}(y-\lambda_n) e^{-\lambda_n t}}{\int_{-\lambda_n}^\infty \mathrm{Ai}(z)^2\, dz}\,.$$

4. *(Essay – 20 pts.)* For **ONE** of the three previous problems —

   (a) Classify the partial differential equation as elliptic, parabolic, or hyperbolic. (Remember to make clear **which** of the three equations you're talking about!)

1: elliptic 2: hyperbolic 3: parabolic

   (b) Explain why we care: List some consequences of the classification for the equation you chose. (What does it tell you about proper boundary conditions, smoothness of solutions, ... ?)

[Please see Qu. 6 of the final exam solutions for Fall 2000 for a list of properties of the three types of equation.]

5. *(60 pts.)* Do **TWO** of these. (Extra credit for all three.)

   (A) For a Sturm–Liouville problem of the form

$$\frac{d}{dx}\left[p(x)\,\frac{d\phi}{dx}\right] + q(x)\phi = -\lambda\phi\,, \qquad \phi'(a) = \gamma\phi(a)\,, \quad \phi(b) = 0$$

   (with $a < x < b$ and $p(x) > 0$), find conditions on q and γ that guarantee that all the eigenvalues λ are positive. *Hint:* Integrate by parts; the weight function is $r(x) = 1$.

Assume that ϕ is a solution. Then

$$\begin{aligned}
\lambda \int_a^b |\phi(x)|^2\, dx &= -\int_a^b \phi(x)^* \left\{ \frac{d}{dx}\left[p(x)\,\frac{d\phi}{dx}\right] + q(x)\phi \right\} dx \\
&= -\phi(b)^* p(b)\phi'(b) + \phi(a)^* p(a)\phi'(a) + \int_a^b p(x)\phi'(x)^*\phi'(x)\, dx - \int_a^b q(x)|\phi(x)|^2\, dx \\
&= \gamma p(a)|\phi(a)|^2 + \int_a^b p(x)|\phi'(x)|^2\, dx - \int_a^b q(x)|\phi(x)|^2\, dx\,.
\end{aligned}$$

Thus λ will be positive if all the terms on the right are nonnegative and at least one of them is positive. This is guaranteed if

$$\gamma \ge 0, \qquad q(x) \le 0,$$

and $\phi'(x)$ is not identically zero. (If this last condition fails — i.e., ϕ is a constant — then λ could be 0. That actually happens for the Neumann problem leading to Fourier cosine series, as we well know.)

(B)

(a) What is the Parseval equation (the formula for $\sum_{n=1}^{\infty} |b_n|^2$) associated with the Fourier sine series on the interval $(0,\pi)$? You probably don't remember the numerical coefficient, so you should rederive it from the orthogonality relation of the functions $\{\sin(nx)\}$.

The series is

$$f(x) = \sum_{n=0}^{\infty} b_n \sin(nx)\,.$$

The orthogonality relation is

$$\int_0^{\pi} \sin(nx)\sin(mx)\,dx = \frac{\pi}{2}\,\delta_{nm}\,.$$

Therefore, multiplying two copies of the series and integrating, we get

$$\int_0^{\pi} |f(x)|^2\,dx = \frac{\pi}{2}\sum_{n=0}^{\infty} |b_n|^2\,.$$

(b) Given (**don't** rederive it!) that the Fourier sine series of $f(x) = x$ on $(0,\pi)$ is

$$x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n}\sin(nx)\,,$$

use the Parseval equation to prove that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$.

In this case we have

$$\int_0^{\pi} x^2\,dx = \tfrac{1}{3}x^3\Big|_0^{\pi} = \frac{\pi^3}{3}\,,$$

and from (a) this equals

$$\frac{\pi}{2}\sum_{n=1}^{\infty}\frac{4}{n^2}\,.$$

Dividing by 2π we get the desired equation. (This famous number is called $\zeta(2)$, ζ being the Riemann zeta function.)

(C) Solve the wave problem (Qu. 2) by the other method. (If you used Fourier analysis before, use d'Alembert's method now; and vice versa.)