Math. 412 (Fulling) 16 November 2012

Test C – Solutions

Calculators may be used for simple arithmetic operations only!

Useful information:

Laplacian operator in cylindrical coordinates:

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}\,.$$

Bessel's equation:

$$\frac{\partial^2 Z}{\partial z^2} + \frac{1}{z}\frac{\partial Z}{\partial z} + \left(1 - \frac{n^2}{z^2}\right) Z = 0 \quad \text{has solutions } J_n(z) \text{ and } Y_n(z)\,.$$

1. *(40 pts.)* Solve as completely as you can:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad (0 < x < L\,, \quad -\infty < t < \infty),$$

$$u(0,t) = 0\,, \quad \frac{\partial u}{\partial x}(L,t) + \gamma u(L,t) = 0\,,$$

$$u(x,0) = f(x)\,, \quad \frac{\partial u}{\partial t}(x,0) = g(x)\,.$$

(Assume that all the eigenvalues you encounter (for $X'' = -\lambda X$) are positive.)

Separating variables as $u_{\text{sep}} = T(t)X(x)$ yields

$$T'' = -\lambda T\,, \quad X'' = -\lambda X\,, \quad X(0) = 0\,, \quad X'(L) + \gamma X(L) = 0\,.$$

If we assume $\lambda > 0$ and set $\lambda = \omega^2$, we get $X(x) = \sin(\omega x)$ (because of the first boundary condition) and $\omega\cos(\omega L) + \gamma\sin(\omega L) = 0$ (because of the second one). Write the eigenvalue condition as

$$\tan(\omega L) = -\frac{\omega}{\gamma}$$

and sketch the graphs of the two sides of that equation. We see that each branch of the tangent curve except the first intersects the straight line once, yielding a root ω_n slighly greater than $\frac{\gamma\pi}{2L}(2n-1)$ (and increasingly close to that value as n becomes large). Then the general solution of the wave equation is

$$u(x,t) = \sum_{n=1}^{\infty} \Big[a_n \sin(\omega_n x)\cos(\omega_n t) + b_n \sin(\omega_n x)\sin(\omega_n t)\Big]\,.$$

We will need the normalization constant

$$N_n = \int_0^L \sin(\omega_n x)^2 \, dx$$

(which can be evaluated in terms of ω_n, but I won't). Then

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(\omega_n x)$$

with

$$a_n = \frac{1}{N_n} \int_0^L \sin(\omega_n x) f(x) \, dx$$

and

$$g(x) = \sum_{n=1}^{\infty} \omega_n b_n \sin(\omega_n x) \cos(\omega_n t)$$

with

$$b_n = \frac{1}{\omega_n N_n} \int_0^L \sin(\omega_n x) g(x) \, dx \,.$$

2. *(10 pts.) (continuation of Question 1)*

regular: Prove that all the eigenvalues are indeed positive, if $\gamma > 0$.
Hint: Write out $\langle X, X'' \rangle$, where $\langle \cdot, \cdot \rangle$ is the inner product appropriate to the problem, and integrate by parts.

On the one hand,

$$\langle X, X'' \rangle = -\lambda \langle X, X \rangle = -\lambda \int_0^L X(x)^2 \, dx \,.$$

On the other hand,

$$\langle X, X'' \rangle = \int_0^L X(x) X''(x) \, dx = X(x) X'(x) \Big|_0^L - \int_0^L X'(x) X'(x) \, dx = -\gamma X(L)^2 - \int_0^L X'(x)^2 \, dx \,.$$

Put it all together and cancel a sign:

$$\lambda \int_0^L X(x)^2 \, dx = \gamma X(L)^2 + \int_0^L X'(x)^2 \, dx \,.$$

Since all the squares are positive, λ must be positive.

honors: Determine what happens when $\gamma < 0$.

When γ is negative but small ($L\gamma > -1$), the straight line intersects the first branch of the tangent curve also. When $L\gamma = -1$, this root hits 0; in that case the eigenfunction is $X(x) = x$. We therefore suspect that for $L\gamma < -1$ there is a negative eigenvalue. So, we try $\lambda = -\rho^2$ and get $X(x) = \sinh(\rho x)$ with $\rho\cosh(\rho L) + \gamma\sinh(\rho L) = 0$, or

$$\tanh(\rho L) = -\frac{\rho}{\gamma}.$$

The graph of $\tanh(\rho L)$ starts at the origin with slope L and has a horizontal asymptote at height 1. Therefore, it intersects the line (once) if and only if $\frac{1}{|\gamma|} < L$, or $L\gamma < -1$, as expected. (Intersections at zero and negative values of ρ are not relevant.) In this situation the solution of the wave equation includes terms

$$a_0\sinh(\rho x)\cosh(\rho t) + b_0\sinh(\rho x)\sinh(\rho t)$$

with

$$a_0 = \frac{1}{N_0}\int_0^L \sinh(\rho x) f(x)\,dx,$$

$$b_0 = \frac{1}{\rho N_0}\int_0^L \sinh(\rho x) g(x)\,dx,$$

$$N_0 = \int_0^L \sinh^2(\rho x)\,dx.$$

3. *(50 pts.)* Solve the 3-dimensional Laplace equation in a cylinder,

$$\nabla^2 u = 0, \qquad (0 < z < L, \quad 0 \le r < S, \quad 0 \le \theta < 2\pi),$$

$$u(S,\theta,z) = 0, \quad u(r,\theta,0) = 0, \quad u(r,\theta,L) = h(r,\theta),$$

periodic boundary conditions in θ.

Separate variables as $u_{\text{sep}} = R(r)\Theta(\theta)Z(z)$, getting

$$0 = R''\Theta Z + \frac{1}{r}R'\Theta Z + \frac{1}{r^2}R\Theta'' Z + R\Theta Z''.$$

We can foresee that $\Theta(\theta) = e^{in\theta}$ for integer n, and that Z''/Z will equal another separation constant. Because the homogeneous boundary condition is on R, not Z, one can anticipate that this constant will be positive, so $Z(z) = \sinh(\omega z)$ and

$$0 = R'' + \frac{1}{r}R' - \frac{n^2}{r^2}R + \omega^2 R$$

with $R(S) = 0$ and R regular at the axis of the cylinder. Thus $R(r) = J_n(\omega r)$ with $J_n(\omega S) = 0$. Let z_{nj} be the j th root of J_n. The complete solution is now

$$u(r,\theta,z) = \sum_{n=-\infty}^{\infty}\sum_{j=1}^{\infty} C_{nj} J_n\left(\frac{z_{nj}r}{S}\right) e^{in\theta}\sinh\left(\frac{z_{nj}z}{S}\right).$$

It remains to satisfy

$$h(r,\theta) = \sum_{n=-\infty}^{\infty} \sum_{j=1}^{\infty} C_{nj} J_n \left(\frac{z_{nj} r}{S}\right) e^{in\theta} \sinh \left(\frac{z_{nj} L}{S}\right) .$$

Using orthogonality of the 2-dimensional eigenfunctions and the known normalization of the $\{e^{in\theta}\}$, we find

$$C_{nj} \sinh \left(\frac{z_{nj} L}{S}\right) = \frac{\int_0^{2\pi} d\theta \int_0^S r\, dr\, e^{-in\theta} J_n\left(\frac{z_{nj} r}{S}\right) h(r,\theta)}{2\pi \int_0^S J_n\left(\frac{z_{nj} r}{S}\right)^2 r\, dr} .$$