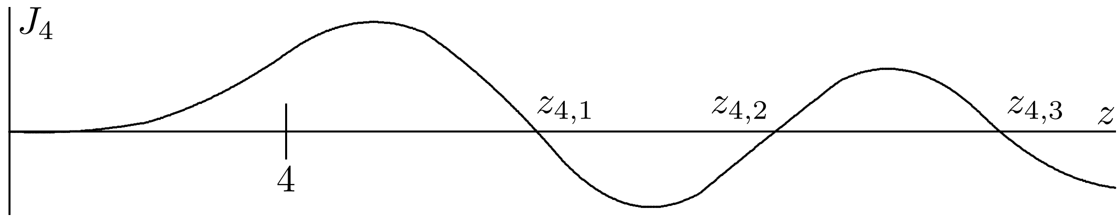
Finishing up the drum problem

Recall that we were seeking normal modes $\phi_\nu(r,\theta) = R(r)\Theta(\theta)$, where

$$\Theta(\theta) = e^{i\nu\theta} \equiv e^{\pm in\theta} \quad \text{with } \nu \text{ an integer}$$

and $R(r) = Z_n(\omega r)$ had to be a Bessel function satisfying appropriate boundary conditions at the origin and the edge of the disk ($r = r_0$). As we have seen, the condition that R remain bounded as $r \to 0$ implies that Z_n is (a multiple of) J_n. The other condition is that $R(r_0) = 0$. It follows that ωr_0 must equal a *zero* (*root*) of $J_n(z)$. Since J_n eventually becomes oscillatory, there are infinitely many such values of z; let us call them $z_{n1}, z_{n2}, \ldots$. They can be found by numerical methods and are tabulated in many handbooks. They are the analogue for the current problem of the numbers $n\pi/L$ in simple Fourier problems and the roots of the equations such as $\tan z = -\gamma z$ in convective Sturm–Liouville problems.



Therefore, the eigenvalues of our problem (or, rather, their square roots) are

$$\omega_{nk} \equiv \frac{z_{nk}}{r_0}\,. \tag{1}$$

The presence of ω_{nk} scaling the radial coordinate "compresses" the nth Bessel function so that k of the lobes of its graph fit inside the disk of radius r_0. Putting the radial and angular parts together, we have the eigenfunctions

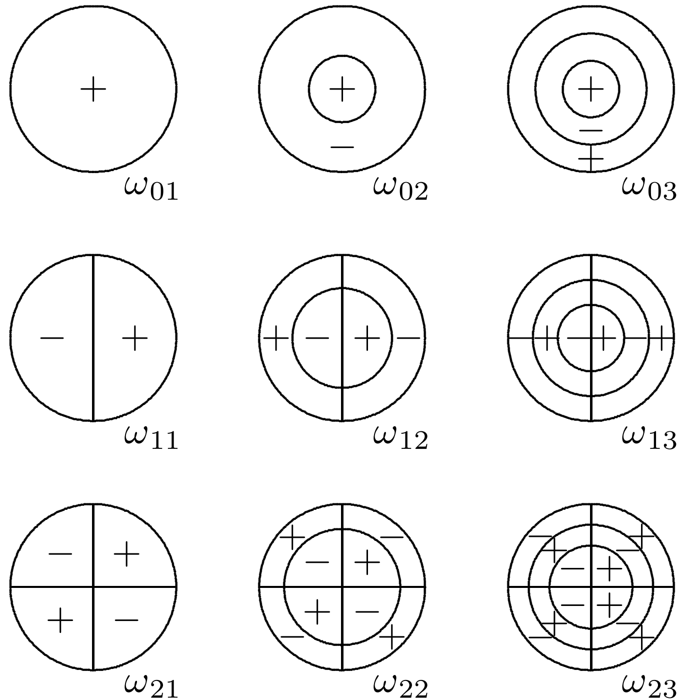
$$\psi_{\nu k}(r,\theta) = R(r)\Theta(\theta) = J_n(\omega_{nk} r)\, e^{i\nu\theta} \qquad (\nu = \pm n). \tag{2}$$

We could equally well use the *real* eigenfunctions in which $e^{i\nu\theta}$ is replaced by $\sin n\theta$ or $\cos n\theta$; those functions are easier to visualize. In the drawing the lines and curves indicate places where such a ψ equals 0, and the signs indicate how the solution $\operatorname{Re}\psi$ or $\operatorname{Im}\psi$ bulges above or below the plane $\psi = 0$. Such patterns may be seen in the surface of a cupful of coffee or other liquid when the container is tapped lightly. (Compare the rectangle eigenfunctions in an earlier section.)

If we were solving a heat-conduction problem in the disk, the general solution would be a linear combination of the separated solutions:

$$u(t, r, \theta) = \sum_{\nu=-\infty}^{\infty} \sum_{k=1}^{\infty} c_{\nu k}\, \psi_{\nu k}(r, \theta)\, e^{-\omega_{nk}{}^2 t}. \tag{3}$$

The coefficients need to be calculated from the initial data:


ω_{01}
ω_{02}
ω_{03}
ω_{11}
ω_{12}
ω_{13}
ω_{21}
ω_{22}
ω_{23}

$$
\begin{aligned}
g(r,\theta) &= u(0,r,\theta) \\
&= \sum_{\nu=-\infty}^{\infty} \sum_{k=1}^{\infty} c_{\nu k}\, \psi_{\nu k}(r,\theta) \\
&= \sum_{\nu=-\infty}^{\infty} \sum_{k=1}^{\infty} c_{\nu k}\, J_n(\omega_{nk} r)\, e^{i\nu\theta}.
\end{aligned}
$$

By the standard Fourier series formula,

$$
\frac{1}{2\pi} \int_0^{2\pi} e^{-i\nu\theta}\, g(r,\theta)\, d\theta = \sum_{k=1}^{\infty} c_{\nu k}\, J_n(\omega_{nk} r).
$$

We are left with a one-dimensional series in the eigenfunctions $R_n(r) \equiv J_n(\omega_{nk} r)$ (n fixed). We recall that these functions came out of the equation

$$
R'' + \frac{1}{r}\, R' + \left(\omega^2 - \frac{n^2}{r^2}\right) R = 0
$$

with the boundary condition $R(r_0) = 0$, which looks like a Sturm–Liouville problem. Unfortunately, it does not quite satisfy the technical conditions of the Sturm–Liouville theorem, because of the singular point in the ODE at $r = 0$. Nevertheless, it turns out that the conclusions of the theorem are still valid in this case: The eigenfunctions are complete (for each fixed n), and they are orthogonal with respect to the weight function r:

$$\int_0^{r_0} J_n(\omega_{ni} r)\, J_n(\omega_{nj} r)\, r\, dr = 0 \quad \text{if } i \neq j.$$

Thus if $h(r)$ is an arbitrary function on $[0, r_0]$, it can be expanded as

$$h(r) = \sum_{k=1}^{\infty} c_k\, J_n(\omega_{nk} r),$$

and the coefficients are

$$c_k = \frac{\int_0^{r_0} J_n(\omega_{nk} r)\, h(r)\, r\, dr}{\int_0^{r_0} J_n(\omega_{nk} r)^2\, r\, dr}\,.$$

Furthermore, the experts on Bessel functions assure us that the integral in the denominator can be evaluated:

$$\int_0^1 J_n(z_{nk}\zeta)^2\,\zeta\,d\zeta = \frac{1}{2}\,J_{n+1}(z_{nk})^2.$$

(I leave the change of variable from ζ to r as an exercise.)

Applying this theorem to our problem, we get

$$c_{\nu k} = \left[\frac{r_0^2}{2}\,J_{n+1}(\omega_{nk})^2\right]^{-1} \int_0^{r_0} r\,dr\,J_n(\omega_{nk}r)\,\frac{1}{2\pi}\int_0^{2\pi} d\theta\,e^{-i\nu\theta}\,g(r,\theta). \tag{4}$$

That is,

$$\begin{aligned} c_{\nu k} &= \left[\pi r_0^2 J_{n+1}(\omega_{nk})^2\right]^{-1} \int_{r=0}^{r_0}\int_{\theta=0}^{2\pi} r\,dr\,d\theta\,\psi_{\nu k}(r,\theta)^*\,g(r,\theta) \\ &= \frac{1}{\|\psi_{\nu k}\|^2}\int_{r=0}^{r_0}\int_{\theta=0}^{2\pi} r\,dr\,d\theta\,\psi_{\nu k}(r,\theta)^*\,g(r,\theta). \end{aligned}$$

(In the last version I have identified the constant factor as the normalization constant for the *two-dimensional* eigenfunction.) We now see that the mysterious weight factor r has a natural geometrical interpretation: It makes the r and θ integrations go together to make up the standard integration over the disc in polar coordinates!

The formulas (1)–(4) give a complete solution of the heat-conduction problem.

But I thought we were solving the wave equation, to model the vibrations of a drum? Yes, your absent-minded professor shifted to the heat equation in mid-stream, then decided to stay there to keep the formulas simpler. What changes are needed in the foregoing to finish the drum problem? The eigenvalues (1) and eigenfunctions (2) are the same. However, for each eigenfunction there are now *two* possible terms in the solution; the eigenfunction expansion (3) needs to be replaced by

$$u(t,r,\theta) = \sum_{\nu=-\infty}^{\infty} \sum_{k=1}^{\infty} c_{\nu k}\, \psi_{\nu k}(r,\theta)\, \cos(\omega_{nk}t) + d_{\nu k}\, \psi_{\nu k}(r,\theta)\, \sin(\omega_{nk}t).$$

[There is an important pitfall to avoid here, which is not confined to polar coordinates. (It also arises, for instance, in the wave equation for vibrations in a ring, using Fourier series.) Suppose that you chose to use the real eigenfunctions. Then it would be a mistake to write in the summand something like

$$[a_{nk}\cos(n\theta) + b_{nk}\sin(n\theta)][c_{nk}\cos(\omega_{nk}t) + d_{nk}\sin(\omega_{nk}t)].$$

This would result in equations for the unknown coefficients that are nonlinear, hence hard to solve; also, the solution will not be unique, and may not even exist for some initial data. Remember to write the general solution as a *linear combination* of all possible (independent) elementary separated solutions:

$$A_{nk}\cos(n\theta)\cos(\omega_{nk}t) + B_{nk}\cos(n\theta)\sin(\omega_{nk}t) + C_{nk}\sin(n\theta)\cos(\omega_{nk}t) + D_{nk}\sin(n\theta)\sin(\omega_{nk}t).$$

In other words, multiply first, *then* superpose!]

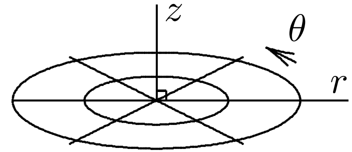
To finish the problem, we need to set u and its time derivative equal to the given initial data and solve for the c and d coefficients. The same orthogonality properties used in the treatment of the heat equation apply here, so (after twice as much work) you will end up with formulas analogous to (4).

A higher-dimensional example

We shall consider the three-dimensional potential equation in a cylinder. (See J. D. Jackson, *Classical Electrodynamics*, Chapter 3.)

Cylindrical coordinates are defined by

$$x = r\cos\theta,$$
$$y = r\sin\theta,$$
$$z = z.$$

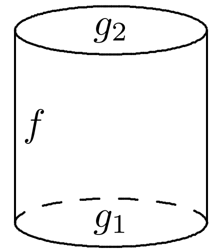


The Laplacian operator is

$$\begin{aligned}\nabla^2 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \\ &= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}.\end{aligned}$$

The problem to solve is: $\nabla^2 u = 0$ inside the cylinder, with Dirichlet data given on all three parts of the cylinder's surface:

$$u(r_0,\theta,z) = f(\theta,z),$$
$$u(r,\theta,0) = g_1(r,\theta),$$
$$u(r,\theta,L) = g_2(r,\theta).$$



As usual, the first step is to split this into two problems:

$$u = u_1 + u_2\,,$$

where

$$u_1(r_0,\theta,z) = 0 \quad \text{with nonhomogeneous data on the end faces,}$$

$$u_2(r,\theta,0) = 0 = u_2(r,\theta,L) \quad \text{with nonhomogeneous data on the curved surface.}$$

In either of these subproblems we can separate variables this way:

$$u = R(r)\Theta(\theta)Z(z).$$

After several routine steps (exercise) we get

$$\frac{d^2Z}{dz^2} - \omega^2 Z = 0, \qquad \frac{d^2\Theta}{d\theta^2} + \mu^2\Theta = 0,$$

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(\omega^2 - \frac{\mu^2}{r^2}\right) R = 0$$

except that it is not yet clear whether the quantities here named ω^2 and μ^2 are really positive. (If we find out they aren't, we'll change notation.) Note that the radial equation is a Bessel equation.

Problem 1: In the u_1 problem the homogeneous boundary condition is $R(r_0) = 0$. The equations determining Θ and R are identical to these we solved in the drum problem. So, we have $\mu = \pm n$, an integer, and then $\omega_{nk} = z_{nk}/r_0$, where z_{nk} is the kth zero of J_n. The new element is the Z equation, whose solutions are exponentials. As in some previous problems, the most convenient basis for these solutions consists of certain hyperbolic functions. Cutting a long story short, we arrive at the general solution

$$\begin{aligned} u_1(r,\theta,z) = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} J_n(\omega_{nk}r)\,[&A_{nk}\cos n\theta\,\sinh\omega_{nk}z + B_{nk}\sin n\theta\,\sinh\omega_{nk}z \\ &+ C_{nk}\cos n\theta\,\sinh\omega_{nk}(L-z) + D_{nk}\sin n\theta\,\sinh\omega_{nk}(L-z)]\,. \end{aligned}$$

(I chose real eigenfunctions for variety, and to reinforce an earlier warning about how to write correct linear combinations of normal modes.) Then, for example, we have

$$\begin{aligned} g_1(r,\theta) &= u_1(r,\theta,0) \\ &= \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} J_n(\omega_{nk}r)\,[C_{nk}\cos(n\theta)\sinh(\omega_{nk}L) + D_{nk}\sin(n\theta)\sinh(\omega_{nk}L)]\,, \end{aligned}$$

and therefore

$$C_{nk} = \frac{\int_0^{r_0} r\,dr \int_0^{2\pi} d\theta\, J_n(\omega_{nk} r)\ \cos(n\theta)\, g_1(r,\theta)}{\pi \sinh(\omega_{nk} L) \int_0^{r_0} J_n(\omega_{nk} r)^2\, r\, dr}.$$

The solutions for C_{0k} and D_{nk}, and the solutions for A_{nk} and B_{nk} in terms of g_2, are similar (and by now routine).

It is interesting to vary this problem by taking the radius r_0 to infinity — in other words, solving Laplace's equation in the whole plane, described in polar coordinates. Then the series of Bessel functions goes over into an integral transform, analogous to the Fourier transform. The initial-data formulas above become

$$g_1(r,\theta) = \sum_{n=0}^{\infty} \int_0^{\infty} d\omega\, J_n(\omega r)\, [C_n(\omega) \cos(n\theta) \sinh{(\omega L)} + D_n(\omega) \sin(n\theta) \sinh{(\omega L)}]\,,$$

$$C_n(\omega) = \frac{\omega}{\pi} \int_0^\infty r\,dr \int_0^{2\pi} d\theta\, J_n(\omega r)\, \cos(n\theta)\, g_1(r,\theta).$$

(This is not supposed to be obvious; proving it is beyond the scope of this course.)

To clarify the crux of these Bessel expansions, let's strip away the angular complications and summarize them as one-dimensional eigenfunction expansions. Consider an arbitrary function $f(r)$.

1. **Fourier–Bessel series:** If the domain of f is $0 < r < r_0$, and $\omega_{nk} \equiv z_{nk}/r_0$, then (for a *fixed* n)

   $$f(r) = \sum_{k=1}^{\infty} A_k\, J_n(\omega_{nk} r),$$

   where

   $$A_k = \frac{\int_0^{r_0} J_n(\omega_{nk} r)\, f(r)\, r\,dr}{\int_0^{r_0} J_n(\omega_{nk} r)^2\, r\,dr}\,.$$

This is a generalization of the Fourier sine series, where the ordinary differential equation involved is a variable-coeffient equation (Bessel's) instead of $X'' = -\omega^2 X$.

2. **Hankel transform:** If the domain of f is $0 < r < \infty$, then (for a fixed n)

$$f(r) = \int_0^\infty A(\omega)\, J_n(\omega r)\, \omega\, d\omega,$$

where

$$A(\omega) = \int_0^\infty f(r)\, J_n(\omega r)\, r\, dr.$$

This is a generalization of the Fourier sine transform.

Problem 2: In the u_2 problem the homogeneous boundary conditions are $Z(0) = 0 = Z(L)$. This goes with the ODE $Z'' - \omega^2 Z = 0$. We see that ω^2 must be negative this time, so we should change the notation accordingly:

$$\omega^2 \equiv -\nu^2 < 0.$$

We can write the solutions as

$$Z(z) = A\cos(\nu z) + B\sin(\nu z), \qquad \nu = \frac{m\pi}{L}\,.$$

From the θ equation (whose boundary conditions are unchanged) we still have $\mu = n$. Therefore, the radial equation is

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(-\nu^2 - \frac{\mu^2}{r^2}\right)R = 0$$

with μ and ν related to integers n and m as just decribed.

The solutions of this equation are *modified Bessel functions*, which are regular Bessel functions evaluated at *imaginary argument*. Letting $\zeta \equiv i\nu r$ puts the equation into standard form:

$$\frac{d^2 R}{d\zeta^2} + \frac{1}{\zeta}\frac{dR}{d\zeta} + \left(1 - \frac{n^2}{\zeta^2}\right) R = 0.$$

Thus R as a function of ζ is a standard Bessel function, so R as a function of r is a modified Bessel function. In the standard notation for modified Bessel functions introduced earlier, R must be a linear combination of $I_n(\nu r)$ and $K_n(\nu r)$, where I is the one that is nice at 0 and K is the one that is nice at infinity. In our problem, zero is the relevant boundary, so $R(r) = I_n(\nu r)$ and

$$u_2(r, \theta, z) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \sin \frac{m\pi z}{L} \left[A_{mn} \cos(n\theta) + B_{mn} \sin(n\theta)\right] I_n \left(\frac{m\pi r}{L}\right).$$

Apply the nonhomogeneous boundary condition:

$$
\begin{aligned}
f(\theta, z) &= u_2(r_0, \theta, z) \\
&= \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \sin \frac{m\pi z}{L} \left[A_{mn} \cos(n\theta) + B_{mn} \sin(n\theta) \right] I_n \left(\frac{m\pi r_0}{L} \right),
\end{aligned}
$$

and the coefficients are found by two steps of ordinary Fourier series inversion. In this case the Bessel functions are not used as elements of a basis of eigenfunctions to expand data; rather, they play the same auxiliary role as the $\sinh(\omega L)$ in some of our Cartesian potential problems and the $e^{-\omega^2 t}\big|_{t=0} = 1$ in heat problems.