Bessel functions

The drum problem: Consider the wave equation (with $c = 1$) in a disc with homogeneous Dirichlet boundary conditions:

$$\nabla^2 u = \frac{\partial^2 u}{\partial t^2}, \qquad u(r_0, \theta, t) = 0,$$

$$u(r, \theta, 0) = f(r, \theta), \qquad \frac{\partial u}{\partial t}(r, \theta, 0) = g(r, \theta).$$

(Note that to solve the *nonhomogeneous* Dirichlet problem for the wave equation, we would add this solution to that of the disc potential problem, **I**, solved in the previous section; the latter is the steady-state solution for the wave problem.)

We expect to get a sum over normal modes,

$$u = \sum_n \phi_n(r, \theta) T_n(t).$$

Let us seek the separated solutions: If $u_{\text{sep}} = \phi(r,\theta)T(t)$, then

$$\frac{\nabla^2\phi}{\phi} = \frac{T''}{T} = -\omega^2.$$

Therefore

$$T = \alpha e^{i\omega t} + \beta e^{-i\omega t} = A\cos(\omega t) + B\sin(\omega t).$$

As for ϕ, it will be periodic in θ and satisfy $\phi(r_0,\theta) = 0$ along with the *Helmholtz equation*

$$-\omega^2\phi = \nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}.$$

This is still a partial DE, so we separate variables again: $\phi = R(r)\Theta(\theta)$,

$$\frac{r(rR')'}{R} + r^2\omega^2 = -\frac{\Theta''}{\Theta} = \nu^2.$$

(In the last section we did this step for $\omega = 0$, and ν^2 was called $-K$.) The boundary condition becomes $R(r_0) = 0$, and as in the previous disc problem we need to assume that R is bounded as $r \to 0$, so that ϕ will be differentiable at the origin and be a solution there. The angular equation is the familiar $\Theta'' = -\nu^2\Theta$, with solutions

$$\Theta(\theta) = e^{\pm in\theta} \quad \text{with } n = |\nu| \text{ an integer.}$$

Remark: Unlike the last disc problem, here we have homogeneous BC on *both* Θ and R. The nonhomogeneity in this problem is the initial data on u.

We can write the radial equation in the Sturm–Liouville form

$$(rR')' - \frac{n^2}{r} R + \omega^2 rR = 0$$

or in the form

$$R'' + \frac{1}{r} R' + \left(\omega^2 - \frac{\nu^2}{r^2}\right) R = 0.$$

This is called *Bessel's equation* if $\omega^2 \neq 0$. (We already studied the case $\omega = 0$ at length. Recall that the solutions were powers of r, except that $\ln r$ also appeared if $n = 0$.) We can put the Bessel equation into a standard form by letting

$$z \equiv \omega r; \qquad r = \frac{z}{\omega}, \qquad \frac{d}{dr} = \omega \frac{d}{dz}.$$

After dividing by ω^2 we get

$$\boxed{\frac{d^2 R}{dz^2} + \frac{1}{z}\frac{dR}{dz} + \left(1 - \frac{n^2}{z^2}\right) R = 0.}$$

(The point of this variable change is to get an equation involving only one arbitrary parameter instead of two.)

If we have a solution of this equation, say $R = Z_n(z)$, then $R(r) \equiv Z_n(\omega r)$ is a solution of the original equation (with $\nu = \pm n$). All solutions $Z_n(z)$ are called

Bessel functions of order n. Although they are not expressible in terms of elementary functions (except when n is half an odd integer), they have been studied so much that many properties of them are known and tabulated in handbooks, symbolic algebra programs, etc.

Remark: For the disk problem, n must be an integer (which we can take nonnegative), but for *sector* problems, other values of n can appear.

Properties of Bessel functions

Power series solution:

In a good differential equations course one learns to substitute

$$Z_n(z) = z^\alpha \sum_{m=0}^{\infty} c_m z^m$$

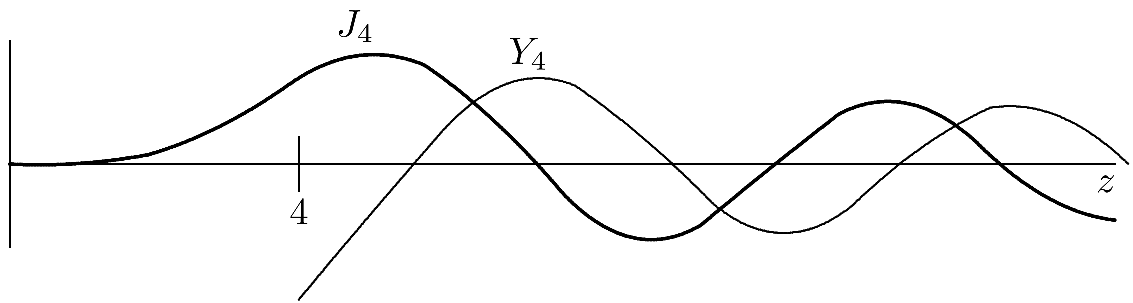
into Bessel's equation, equate the coefficient of each power of z to 0, and try to solve for α and the c_m ("method of Frobenius"). It turns out that $\alpha = \pm n$ (so α can be identified with the ν of the original equation), and that for n a nonnegative integer there is a solution of the assumed form only for the positive root. It is called "J":

$$J_n(z) \equiv \left(\frac{z}{2}\right)^n \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,(n+m)!} \left(\frac{z}{2}\right)^{2m} .$$

This series converges for all z.

Any solution linearly independent of J_n has a singularity at $z = 0$ (in fact, it goes to ∞ in absolute value there). For noninteger n the series with $\alpha = -n$ exists and contains negative powers, but for integer n the second solution involves a logarithm. (It can be found by the method of "reduction of order".) This second solution is nonunique, because of the freedom to multiply by a constant and the freedom to add a multiple of J_n. However, there is a standard choice ("normalization") of the second solution, called either $Y_n(z)$ or $N_n(z)$; I prefer "Y".

General behavior: Here is a graph of J_4 and Y_4. Near the origin, J_n behaves like z^n, while Y_n blows up like z^{-n} (like $\ln z$ if $n = 0$). At large z both functions oscillate, with a slowly decreasing amplitude.



Behavior at small argument ($z \to 0$):

Think of J_n as like r^n, Y_n as like r^{-n}. More precisely,

$$J_n(z) \approx \frac{1}{n!}\left(\frac{z}{2}\right)^n,$$

$$Y_0(z) \approx \frac{2}{\pi}\ln z,$$

$$Y_n(z) \approx -\frac{(n-1)!}{\pi}\left(\frac{z}{2}\right)^{-n} \qquad \text{if } n > 0.$$

Therefore, for a problem inside a disc only J functions will appear, by the boundedness criterion previously mentioned.

Behavior at large argument ($z \to +\infty$):

Think of J_n as like $\cos$, Y_n as like $\sin$. More precisely,

$$J_n(z) \approx \sqrt{\frac{2}{\pi z}}\cos\left(z - \tfrac{1}{2}n\pi - \tfrac{1}{4}\pi\right),$$

$$Y_n(z) \approx \sqrt{\frac{2}{\pi z}} \sin\left(z - \tfrac{1}{2}n\pi - \tfrac{1}{4}\pi\right).$$

One defines the analogues of complex exponentials:

$$H_n^{(1)}(z) \equiv J_n + iY_n \approx \sqrt{\frac{2}{\pi z}}\,(-i)^{n+\frac{1}{2}}\,e^{iz},$$

$$H_n^{(2)}(z) \equiv J_n - iY_n \approx \sqrt{\frac{2}{\pi z}}\,i^{n+\frac{1}{2}}\,e^{-iz}.$$

The crossover point between the $r^{\pm n}$ behavior and the trigonometric behavior is somewhere close to $z = n$.

It is *not* necessary to memorize all these formulas. You should know:

1. J is bounded and smooth at 0; Y isn't.

2. The Bessel functions (for real n and ω) are oscillatory at infinity. (Note that their "envelope" decreases as $1/\sqrt{z}$, but this is not enough to make them square-integrable.)

Recursion relations:

$$
\begin{aligned}
zJ_n' + nJ_n &= zJ_{n-1}\,, \\
zJ_n' - nJ_n &= -zJ_{n+1}\,.
\end{aligned}
$$

From these follow

$$
\frac{2n}{z}\,J_n = J_{n-1} + J_{n+1}
$$

and, most useful of all,

$$
2J_n' = J_{n-1} - J_{n+1}\,.
$$

(So the *derivative* of a Bessel function is not really a new function. Note that the *second* (and hence any higher) derivative can be calculated using the Bessel equation itself.)

The recursion relations are useful in many ways. For instance, computer programs need to calculate J_n "by brute force" only for a few values of n and then use the recursion relations to interpolate.

Modified Bessel functions (and other such things)

In the application just discussed, we had $\nu^2 > 0$ and $\omega^2 > 0$. But Bessel's equation,

$$R'' + \frac{1}{r} R' + \left(\omega^2 - \frac{\nu^2}{r^2}\right) R = 0.$$

also makes sense, and has applications, when one or both of these parameters is negative or complex, so that ν or ω is complex. Complex ω corresponds to complex z, since $z = \omega r$. In particular, imaginary ω (negative real ω^2) corresponds to evaluation of the Bessel functions on the imaginary axis: $Z_\nu(i|\omega|r)$. This is analogous to the passage from $e^{\pm nx}$ to $e^{\pm inx}$, which yields the trigonometric functions (except that here we are moving in the reverse direction, as we shall now see).

These Bessel functions of imaginary argument (but real ν) are called *modified Bessel functions*. A standard basis consists of two functions called $I_\nu(z)$ and $K_\nu(z)$, chosen to behave somewhat like $\sinh z$ and e^{-z}, respectively.

Definitions:

$$I_\nu(z) \equiv i^{-\nu} J_\nu(iz),$$
$$K_\nu(z) \equiv \frac{\pi}{2}\, i^{\nu+1} H_\nu^{(1)}(iz).$$

Behavior at small argument ($z \to 0$):

$$
\begin{aligned}
I_\nu(z) &\approx \frac{1}{\nu!}\left(\frac{z}{2}\right)^\nu , \\
K_\nu(z) &\approx \frac{1}{2}\,(\nu-1)!\left(\frac{z}{2}\right)^{-\nu} , \\
K_0(z) &\approx -\ln z.
\end{aligned}
$$

Behavior at large argument ($z \to +\infty$):

$$
\begin{aligned}
I_\nu(z) &\approx \frac{e^z}{\sqrt{2\pi z}} , \\
K_\nu(z) &\approx \sqrt{\frac{\pi}{2z}}\, e^{-z}.
\end{aligned}
$$

In summary, I_ν is designed to vanish at 0, whereas K_ν is designed to vanish at infinity. (But the arbitrary constant factors in the definitions arose by historical accidents that are not worth wondering about.)

An application of modified Bessel functions will be given later.

Bessel functions of imaginary *order* ν appear in separation of variables in *hyperbolic coordinates*,

$$\begin{aligned} t &= r\sinh\theta, \\ x &= r\cosh\theta \end{aligned} \qquad \text{or} \qquad \begin{aligned} t &= r\cosh\theta, \\ x &= r\sinh\theta. \end{aligned}$$

(The first of these transformations of variables can be related to the "twin paradox" in special relativity. The two apply to different regions of the t–x plane.) If you apply such a transformation to the Klein–Gordon equation,

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + m^2 u = 0,$$

you will get for the r dependence a Bessel equation with imaginary ν and real or imaginary ω (depending on which of the two hyperbolic transformations you're using). Therefore, the solutions will be either $J_{i\kappa}$ or $K_{i\kappa}$ functions.

Many ordinary differential equations are Bessel's equation in disguise. That is, they become Bessel's equation after a change of dependent or independent variable, or both. One example is the deceptively simple-looking equation

$$\frac{d^2u}{dx^2} + xu = 0,$$

whose solutions are called *Airy functions*. If you let

$$y \equiv \tfrac{2}{3}x^{\frac{3}{2}}, \qquad u \equiv x^{\frac{1}{2}}Z,$$

then you get

$$\frac{d^2Z}{dy^2} + \frac{1}{y}\frac{dZ}{dy} + \left(1 - \frac{1}{9y^2}\right)Z = 0,$$

the Bessel equation of order $\nu = \frac{1}{3}$. Therefore, the Airy functions are essentially Bessel functions:

$$u = \sqrt{x}\,Z_{\frac{1}{3}}\left(\tfrac{2}{3}x^{\frac{3}{2}}\right).$$