How Fourier transforms interact with derivatives

Theorem: If the Fourier transform of f' is defined (for instance, if f' is in one of the spaces L^1 or L^2, so that one of the convergence theorems stated above will apply), then the Fourier transform of f' is ik times that of f.

This can be seen either by differentiating

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k)\, e^{ikx}\, dk$$

with respect to x, or by integrating by parts in

$$\widehat{f'}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x)\, e^{-ikx}\, dx$$

(at least if we assume that $f(x) \to 0$ as $x \to \pm\infty$).

Similarly, differentiation with respect to k corresponds to multiplication by $-ix$.

Corollary: If $f'(x)$ exists and is in L^2, then $k\,\hat{f}(k)$ is in L^2, and conversely, and similarly with x and k interchanged.

This is another instance of the principle that smoothness properties of f correspond to various degrees of rapid decrease of $\hat{f}$ at ∞, and vice versa.

This differentiation property of Fourier transforms can be used to solve linear differential equations with constant coefficients. Consider the inhomogeneous ordinary differential equation

$$\frac{d^2 f}{dx^2} - \lambda^2 f(x) = g(x),$$

where $\lambda^2 > 0$ and g is square-integrable and piecewise continuous. Take the

Fourier transform of both sides:

$$(ik)^2 \hat{f}(k) - \lambda^2 \hat{f}(k) = \hat{g}(k).$$

Therefore,

$$\hat{f}(k) = \frac{-\hat{g}(k)}{k^2 + \lambda^2},$$

and hence

$$f(x) = \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\hat{g}(k)e^{ikx}}{k^2 + \lambda^2}\, dk.$$

(We won't know how to evaluate this integral, or even whether it can be done analytically, until we know what g is. Nevertheless, this is a definite formula for the solution. We'll return to this formula and press it a little farther later.)

What happened here is that the Fourier transformation converted a differential equation into an algebraic equation, which could be solved by elementary means. Our use of Fourier transforms (and Fourier series) to solve PDEs is really just an extension of this idea.

Once again, another way of looking at the calculation we have just done is as an analogue of *diagonalizing a matrix*. Suppose we want to solve the equation $M\vec{x} = \vec{y}$, where $\vec{x}$ and $\vec{y}$ are in $\mathbf{R}^2$ and M is a 2×2 matrix. If we can find a matrix U for which

$$M = U^{-1}DU, \qquad D = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix},$$

then

$$M^{-1} = U^{-1}D^{-1}U, \qquad D^{-1} = \begin{pmatrix} \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{pmatrix}.$$

Then it is trivial to calculate $\vec{y} = M^{-1}\vec{x}$. In our ODE, the analogue of M is the differential operator $\frac{d^2}{dx^2} - \lambda^2$ and the analogue of U is the Fourier transformation.

We are using the fact that the functions e^{ikx} are *eigenvectors* of the differentiation operation $\frac{d}{dx}$, and hence of M.

You may (should!) object that the general solution of this ODE should contain two arbitrary constants. Indeed, the solution we have found is not the most general one, but it is the only *square-integrable* one. (You can easily check that none of the solutions of the *associated homogeneous equation*,

$$\frac{d^2 f}{dx^2} - \lambda^2 f(x) = 0$$

(with $\lambda^2 > 0$), are square-integrable, so adding one of them to our solution will give a solution of the inhomogeneous equation that is not in L^2.) The Fourier calculation in effect takes place entirely within the vector space $L^2(-\infty, \infty)$ (although the eigenfunctions are not themselves members of that space).

Relation to the Laplace transform

The foregoing may have reminded you of the Laplace-transform technique for solving ODEs. In fact, the two transforms are closely related.

Suppose $f(x) = 0$ for $x < 0$. Then

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(x)\, e^{-ikx}\, dx.$$

Recall that the *Laplace transform* of f is

$$F(s) = \int_0^\infty f(x)\, e^{-sx}\, dx.$$

Allow s to be complex:

$$s = \sigma + ik, \qquad \sigma \text{ and } k \text{ real.}$$

Then

$$\begin{aligned} F(s) &= \int_0^\infty f(x)\, e^{-\sigma x}\, e^{-ikx}\, dx \\ &= \sqrt{2\pi} \times \text{Fourier transform of } f(x)e^{-\sigma x} \qquad\qquad (\sigma \text{ fixed}). \end{aligned}$$

For "most" f's, $f(x)e^{-\sigma x}$ will be square-integrable if σ is sufficiently large, even if f itself is not square-integrable (e.g., $f =$ polynomial for $x > 0$). To attain this result it was crucial that we cut f off below $x = 0$; when we multiply by $e^{-\sigma x}$, $\sigma > 0$, what we gain at $x = +\infty$ we lose at $-\infty$. The Laplace transform (with *time* in the role of x) is useful for solving initial-value problems, where the data and solution functions may not fall off to 0 as the time approaches $+\infty$, but negative values of time are not of interest. (In particular, the Laplace transform with respect to time can be applied to nonhomogeneous boundary data that depend on time, so that the steady-state solution technique does not apply.)

The Fourier inversion formula for $fe^{-\sigma x}$ says

$$f(x)e^{-\sigma x} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\sigma + ik)\, e^{ikx}\, dk,$$

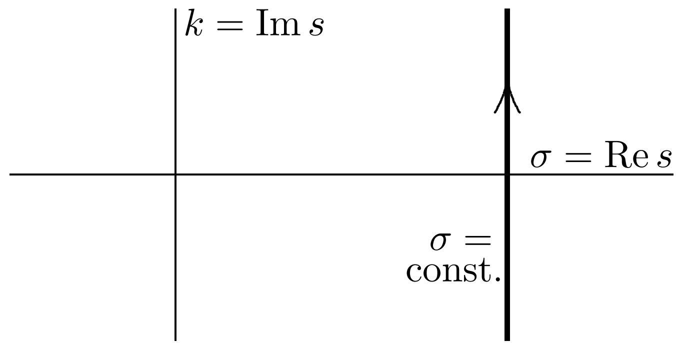
or

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\sigma + ik)\, e^{(\sigma+ik)x}\, dk.$$

In the exponent we recognize the complex variable $s \equiv \sigma + ik$. If we do a formal integration by substitution, taking $ds = i\, dk$, we get

$$\boxed{f(x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(s)\, e^{sx}\, ds.}$$

In courses on complex analysis (such as Math. 407 and 601), it is shown that this integral makes sense as a line integral in the complex plane. It provides an *inversion formula for Laplace transforms*. In elementary differential-equations courses (such as Math. 308) no such formula was available; the only way to invert a Laplace transform was to "find it in the right-hand column of the table" — that is, to know beforehand that that function can be obtained as the direct Laplace transform of something else. The complex analysis courses also provide techniques for evaluating such integrals, so the number of problems that can be solved exactly by Laplace transforms is significantly extended.



In short, the Laplace transform is really the Fourier transform, extended to complex values of k and then rewritten in a notation that avoids complex numbers — until you want a formula to calculate the inverse transform, whereupon the complex numbers come back with a vengeance.

Convolutions, autocorrelation function, and power spectrum

In this course we emphasize the use of the Fourier transform in solving partial differential equations. The Fourier transform also has important applications in signal processing and the analysis of data given as a function of a time variable. Here we take a quick look at some of the tools of that trade.

The Fourier transform of a product of functions is *not* the product of their Fourier transforms! Instead, it is easy to show that that transform is a certain integral involving the transforms of the two factors. This fact is most often used in the inverse direction, so that is how I'll state the formula:

Convolution Theorem: The inverse Fourier transform of $\hat{f}_1(k)\hat{f}_2(k)$ is

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(u) f_2(x-u)\, du \equiv f_1 * f_2\,.$$

This integral is called the *convolution* of f_1 and f_2. Note that

$$f_1 * f_2 = f_2 * f_1\,,$$

although that is not immediately visible from the integral formula.

By manipulating the formulas defining the Fourier transform and its inverse, it is easy to show the following:

Theorem:

(a) If $g(x) \equiv f(-x)$, then $\hat{g}(k) = \hat{f}(-k)$.

(b) if $g(x) \equiv f(-x)$ and $f(x)$ is real-valued, then $\hat{g}(k) = \hat{f}(k)^*$.

Now take $\hat{f}_1 = \hat{f}$ and $\hat{f}_2 = \hat{f}^*$ in the convolution theorem and apply the theorem just stated:

Corollary: If $f(x)$ is real-valued, then the Fourier transform of $|\hat{f}(k)|^2$ is

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) f(u-x)\, du.$$

This integral is called the *autocorrelation function* of f, because it measures to what extent values of f at arguments displaced a distance x tend to coincide. The function $|\hat{f}(k)|^2$ is called the *power spectrum* of f; it measures the extent to which the signal in f is concentrated at frequency k.

As an application of the convolution theorem, return to the differential equation $f'' - \lambda^2 f = g$ and the solution

$$f(x) = \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\hat{g}(k) e^{ikx}}{k^2 + \lambda^2}\, dk.$$

Suppose we knew that

$$-\frac{1}{k^2 + \lambda^2} = \hat{h}(k)$$

for some particular $h(x)$. Then we could write

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(x - t) g(t)\, dt$$

— thereby expressing the solution as a *single* integral instead of two (one to find $\hat{g}$ and then one to find f).

Can we find h? Well, the most obvious way would be to evaluate the inverse Fourier transform,

$$h(x) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{ikx}}{k^2 + \lambda^2} \, dk.$$

Unfortunately, one needs some theorems of complex analysis to evaluate this. Fortunately, I know the answer:

$$h(x) = -\frac{1}{\lambda} \sqrt{\frac{\pi}{2}} \, e^{-\lambda |x|} \quad (\text{if } \lambda > 0).$$

It can be verified by elementary means (see the next section) that this h satisfies

$$\hat{h}(k) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(x) \, e^{-ikx} \, dx = -\frac{1}{k^2 + \lambda^2} \, .$$

So we end up with

$$f(x) = -\frac{1}{2\lambda} \int_{-\infty}^{\infty} e^{-\lambda |x-t|} g(t) \, dt. \qquad (*)$$

The function h, by the way, is called a *Green function* for this problem. It plays the same role as the matrix M^{-1} in the two-dimensional algebraic analogue.

Here is a way to check that $(*)$ is correct.

(1) Find the general solution of $f'' - \lambda^2 f = g$ by "variation of parameters" (see your differential equations textbook, or "Example 2" in the discussion of delta and Green functions below). The answer contains two arbitrary constants and some integrals that are beginning to look like $(*)$.

(2) Determine the arbitrary constants by requiring that f be square-integrable. Then combine terms to get exactly $(*)$.

The uncertainty principle

If $f(x)$ is sharply peaked, then $\hat{f}(k)$ must be spread out; if $\hat{f}(k)$ is sharply peaked, then $f(x)$ must be spread out. A precise statement of this principle is:

$$\int_{-\infty}^{\infty} (x - x_0)^2 |f(x)|^2 \, dx \cdot \int_{-\infty}^{\infty} (k - k_0)^2 |\hat{f}(k)|^2 \, dk \geq \frac{1}{4} \|f\|^4$$

for any numbers x_0 and k_0.

The proof appears in many textbooks of quantum mechanics, or in (for example) Dym and McKean, *Fourier Series and Integrals*, pp. 119–120. It uses the Schwarz inequality and the Parseval identity.

In quantum mechanics, when f is a *wave function* (in a unit system with Planck's constant $\hbar = 1$), $|f(x)|^2$ is the probability density for finding a particle at x and $|\hat{f}(k)|^2$ is the probability density for measuring the particle's momentum to be k. The uncertainty principle is the mathematical reason why the position and momentum can't simultaneously be determined to arbitrary accuracy.

There is also a classical interpretation: Let x be time and f an electrical signal. Then $\hat{f}(k)$ is its frequency spectrum. The uncertainty principle says that a pulse of brief duration must be composed of a wide spectrum of different frequencies; or that to qualify as truly monochromatic, a signal or wave must last for a long time.