Green Functions

Here we will look at another example of how Fourier transforms are used in solving boundary-value problems. This time we'll carry the solution a step further, reducing the solution formula to a single integral instead of a double one.

Laplace's equation in the upper half-plane

Let the ranges of the variables be

$$-\infty < x < \infty, \qquad 0 < y < \infty.$$

Consider the equation

PDE: $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

with the boundary data

$$\text{BC:} \qquad u(x,0) = f(x).$$

This equation might arise as the steady-state problem for heat conduction in a *large* plate, where we know the temperature along one edge and want to simplify the problem by ignoring the effects of the other, distant edges. It could also arise in electrical or fluid-dynamical problems.

It turns out that to get a unique solution we must place one more condition on u: it must remain bounded as x or y or both go to infinity. (In fact, it will turn out that usually the solutions go to 0 at ∞.) Excluding solutions that grow at infinity seems to yield the solutions that are most relevant to real physical situations, where the region is actually finite. But it is the mathematics of the partial differential equation that tells us that to make the problem well-posed we do *not* need to *prescribe some arbitrary function* as the limit of u at infinity, as we needed to do in the case of finite boundaries.

Separating variables for this problem at first gives one a feeling of *déjà vu*:

$$u_{\mathrm{sep}}(x,y)=X(x)Y(y)\ \Rightarrow\ 0=X''Y+XY'';$$

$$-\frac{X''}{X}=\lambda=\frac{Y''}{Y};$$

write λ as k^2. The remaining steps, however, are significantly different from the case of the finite rectangle, which we treated earlier.

If $\lambda \neq 0$, the solution of the x equation can be

$$X(x)=e^{ikx},$$

where any k and its negative give the same λ. The condition of boundedness requires that k be *real* but does not further restrict it! Taking $k=0$ yields the only bounded solution with $\lambda=0$. Therefore, we take the X in each separated solution to be e^{ikx} for some real k. The corresponding λ will be positive or zero.

Turning now to the y equation, we see that Y is some linear combination of e^{ky} and e^{-ky}. For boundedness we need the exponent to be negative, so we write

$$Y(y) = e^{-|k|y} \; (= e^{-\sqrt{\lambda}\, y})$$

to get an expression that's valid regardless of whether k is positive or negative.

We are now finished with the homogeneous conditions, so we're ready to superpose the separated solutions. Since k is a continuous variable, "superpose" in this case means "integrate", not "sum":

$$u(x,y) = \int_{-\infty}^{\infty} dk\, c(k)\, e^{ikx} e^{-|k|y}.$$

Here $c(k)$ is an arbitrary function, which plays the same role as the arbitrary coefficients in previous variable separations. The initial condition is

$$f(x) = \int_{-\infty}^{\infty} dk\, c(k)\, e^{ikx}.$$

Comparing with the formula for the inverse Fourier transform, we see that $c(k) = \frac{1}{\sqrt{2\pi}}\,\hat{f}(k)$. That is,

$$c(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx.$$

In other words, the solution can be written

$$u(x,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, \hat{f}(k)\, e^{ikx} e^{-|k|y}.$$

A Green function for Laplace's equation

We can get a simpler expression for u in terms of f by substituting the formula for $\hat{f}$ into the one for u. But to avoid using the letter x to stand for two different things in the same equation, we must first rewrite the definition of the Fourier transform using a different variable:

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz\, e^{-ikz}\, f(z)\,.$$

Then

$$u(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dz\, e^{ik(x-z)} e^{-|k|y}\, f(z).$$

We'll evaluate this multiple integral with the k integral on the inside. (This step requires some technical justification, but that is not part of our syllabus.) The inner integral is

$$\begin{aligned}\int_{-\infty}^{\infty} dk\, e^{ik(x-z)} e^{-|k|y} &= \int_{-\infty}^{0} dk\, e^{ik(x-z)} e^{ky} + \int_{0}^{\infty} dk\, e^{ik(x-z)} e^{-ky} \\ &= \frac{e^{ik(x-z-iy)}}{i(x-z-iy)} \Big|_{-\infty}^{0} + \frac{e^{ik(x-z+iy)}}{i(x-z+iy)} \Big|_{0}^{\infty} \\ &= \frac{1}{i(x-z-iy)} - \frac{1}{i(x-z+iy)} \\ &= \frac{2y}{(x-z)^2+y^2} \,.\end{aligned}$$

Thus

$$u(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} dz\, \frac{y}{(x-z)^2+y^2}\, f(z). \qquad (*)$$

The function

$$G(x-z,y) \equiv \frac{1}{\pi} \frac{y}{(x-z)^2+y^2}$$

is called a *Green function* for the boundary-value problem we started from. It is also called the *kernel* of the *integral operator*

$$u = G(f)$$

defined by $(*)$. The point of $(*)$ is that it gives the solution, u, as a function of the boundary data, f.

In principle, Green functions exist for the boundary-value problems on finite regions which we have solved earlier. However, in those cases the G is given by an infinite *sum* arising from the Fourier *series*, rather than the *integral* which expresses G in a Fourier-transform problem. Typically, such sums are harder to evaluate than the analogous integrals — which is why we have waited until now to introduce Green functions.

The Green function for the *heat* equation on an infinite interval is derived from the Fourier-transform solution in much the same way. To do that we need a basic integral formula, which I'll now derive.

The integral in question is

$$H(x) \equiv \int_{-\infty}^{\infty} e^{ikx}\, e^{-k^2 t}\, dk,$$

where t is positive.

Note first that

$$\frac{d}{dk} e^{-k^2 t} = -2kt\, e^{-k^2 t}.$$

This will allow us to find a differential equation satisfied by H: From the definition we calculate

$$\begin{aligned} H'(x) &= \int_{-\infty}^{\infty} ik\, e^{ikx}\, e^{-k^2 t}\, dk \\ &= \frac{-i}{2t} \int_{-\infty}^{\infty} e^{ikx} \left(\frac{d}{dk} e^{-k^2 t} \right) dk \\ &= \frac{+i}{2t} \int_{-\infty}^{\infty} \left(\frac{d}{dk} e^{ikx} \right) e^{-k^2 t}\, dk \\ &= \frac{-x}{2t} \int_{-\infty}^{\infty} e^{ikx}\, e^{-k^2 t}\, dk \\ &= -\frac{x}{2t}\, H(x). \end{aligned}$$

Thus

$$\frac{H'}{H} = -\frac{x}{2t}\,;$$

$$\ln H = -\frac{x^2}{4t} + \text{const.};$$

$$H = C\, e^{-x^2/4t}.$$

To find the constant we evaluate the integral for $x = 0$:

$$\begin{aligned} C &= H(0) \\ &= \int_{-\infty}^{\infty} e^{-k^2 t}\, dk \\ &= \frac{1}{\sqrt{t}} \int_{-\infty}^{\infty} e^{-q^2}\, dq, \end{aligned}$$

by the substitution $q = k\sqrt{t}$. But it is well known that

$$\int_{-\infty}^{\infty} e^{-q^2}\, dq = \sqrt{\pi},$$

because its square is

$$\iint_{\mathbf{R}^2} e^{-x^2} e^{-y^2}\, dx\, dy = \int_0^{2\pi} \int_0^{\infty} e^{-r^2}\, r\, dr\, d\theta$$
$$= 2\pi \int_0^{\infty} e^{-u}\, \frac{1}{2}\, du$$
$$= \pi.$$

So

$$C = \sqrt{\frac{\pi}{t}}\,.$$

Therefore, we have shown that $H(x)$ is

$$\boxed{\int_{-\infty}^{\infty} e^{ikx}\, e^{-k^2 t}\, dk = \sqrt{\frac{\pi}{t}}\, e^{-x^2/4t}.}$$

Now I leave it as an exercise* to solve the initial-value problem for the heat equation for $x \in (-\infty, \infty)$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}\,, \qquad \text{(PDE)}$$

$$u(0, x) = f(x), \qquad \text{(IC)}$$

in analogy to our two previous Fourier-transform solutions. You should then find that the problem is solved by the Green function

$$G(t, x - z) \equiv \frac{1}{2\pi}\, H(x - z) = \frac{1}{\sqrt{4\pi t}}\, e^{-(x-z)^2/4t}.$$

Note also that the formula in the box is also useful for evaluating similar integrals with the roles of x and k interchanged. (Taking the complex conjugate of the formula, we note that the sign of the i in the exponent doesn't matter at all.)

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* Or peek at Haberman, Sec. 10.2.