Additional Topics on Green Functions

A Green function for the wave equation

It is relatively difficult to work the Fourier-transform solution of the wave equation into a Green-function form, because the integrals are poorly convergent. However, we already have a Green-function solution of the initial-value problem for the wave equation: it is d'Alembert's solution! Recall that (for $c = 1$, $f(x) \equiv u(0,x)$, $g(x) \equiv \frac{\partial u}{\partial t}(0,x)$) the solution is

$$u(t,x) = \frac{1}{2}[f(x+t) + f(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} g(z)\,dz. \tag{1}$$

For simplicity consider only the case $t > 0$. Then (1) can be written

$$\begin{aligned} u(t,x) &= \frac{1}{2}\int_{-\infty}^{\infty} dz\, f(z)[\delta(z-x-t)+\delta(z-x+t)] \\ &+ \frac{1}{2}\int_{-\infty}^{\infty} dz\, g(z)[h(z-x+t)-h(z-x-t)], \end{aligned} \tag{2}$$

where h is the unit step function; recall that it satisfies

$$\delta(w) = \frac{dh(w)}{dw}\,.$$

Now define

$$G(t,x,z) \equiv \frac{1}{2}[h(z-x+t)-h(z-x-t)],$$

so that

$$\frac{\partial G}{\partial t}(t,x,z) = \frac{1}{2}[\delta(z-x+t)+\delta(z-x-t)].$$

Then (2) can be rewritten as

$$u(t,x) = \int_{-\infty}^{\infty} \frac{\partial G}{\partial t}(t,x,z) u(0,z)\,dz + \int_{-\infty}^{\infty} G(t,x,z) \frac{\partial u}{\partial t}(0,z)\,dz.$$

(Although we assumed $t > 0$, this formula also holds for $t < 0$.)

This particular kind of combination of *boundary values and derivatives of the solution and a Green function* is quite characteristic of boundary-value problems for second-order equations. We'll see it again in connection with Laplace's equation.

Green functions for nonhomogeneous problems

For a variety of historical and practical reasons, this course concentrates on *homogeneous* linear PDEs and their (nonhomogeneous) boundary-value problems. From a Green-function point of view, however, nonhomogeneous differential equations are actually more fundamental. We will look briefly at two of these.

The Green function for the heat equation with source

Recall that the solution of the initial-value problem for the homogeneous heat equation is

$$u(t,x)=\int_{-\infty}^{\infty} H(t,x,y)f(y)\,dy \qquad (f(x)\equiv u(0,x)),$$

where

$$H(t,x,y)=\frac{1}{\sqrt{4\pi t}}\,e^{-(x-y)^2/4t}.$$

H could be defined as the solution of the initial-value problem

$$\frac{\partial H}{\partial t}=\frac{\partial^2 H}{\partial x^2}\,, \qquad H(0,x,y)=\delta(x-y). \tag{4}$$

We are now interested in the nonhomogeneous heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + j(t,x) \quad (\text{for } t > 0), \qquad u(0,x) = 0 \tag{5}$$

(where we've imposed the homogeneous initial condition to make the solution unique). In view of our experience with ODEs we might expect the solution to be of the form

$$u(t,x) = \int_{-\infty}^{\infty} dy \int_0^{\infty} ds\, G(t,x;s,y) j(s,y), \tag{6}$$

where G satisfies

$$\frac{\partial G}{\partial t} - \frac{\partial^2 G}{\partial x^2} = \delta(t-s)\delta(x-y), \qquad G(0,x,s,y) = 0 \tag{7}$$

(i.e., the temperature response to a *point source* of heat at position y and time s.) The surprising fact is that G turns out to be essentially the same thing as H.

To see that, consider

$$u(t,x) \equiv \int_{-\infty}^{\infty} dy \int_0^t ds\, H(t-s,x,y) j(s,y).$$

It can be proved that differentiation "under the integral sign" is legitimate here, so let's just calculate

$$\frac{\partial u}{\partial t} = \int_{-\infty}^{\infty} dy\, H(0,x,y) j(t,y) + \int_{-\infty}^{\infty} dy \int_0^t ds\, \frac{\partial H}{\partial t}(t-s,x,y) j(s,y),$$

$$\frac{\partial^2 u}{\partial x^2} = \int_{-\infty}^{\infty} dy \int_0^t ds\, \frac{\partial^2 H}{\partial x^2}(t-s,x,y) j(s,y).$$

Now use (4) to evaluate the first term in $\frac{\partial u}{\partial t}$ *and* to observe that the other term cancels $\frac{\partial^2 u}{\partial x^2}$ when we construct

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \int_{-\infty}^{\infty} dy\, \delta(x-y) j(t,y) = j(t,x).$$

Also, we have $u(0,x) = 0$. So our u solves the problem (5). In other words, the solutions of (5) is (6) with

$$G(t,x;s,y) = \begin{cases} H(t-s,x,y) & \text{if } s \le t, \\ 0 & \text{if } s > t. \end{cases}$$

Put the other way around: The Green function that solves the initial-value problem for the homogeneous heat equation is

$$H(t,x,y) = G(t,x;0,y),$$

where G is the Green function that solves the nonhomogeneous heat equation with homogeneous initial data (and is defined by (7)). This connection between nonhomogeneous and homogeneous Green functions is called *Duhamel's principle* (specifically, for the heat equation, and more loosely, for analogous more general situations).

The previous result for the wave equation is another instance of this principle: It can be shown that

$$\begin{aligned} G_{\text{ret}}(t,x;s,z) &\equiv G(t-s,x,z)h(t-s) \\ &= \tfrac{1}{2}[h(z-x+t-s)-h(z-x-t+s)]h(t-s) \\ &= \tfrac{1}{2}h(t-x)h(t+x) \end{aligned}$$

is a Green function for the nonhomogeneous wave equation, in the sense that

$$u(t,x) = \int_{-\infty}^{\infty} dy \int_{-\infty}^{t} ds\, G_{\text{ret}}(t,x;s,y)f(s,y)$$

satisfies

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial t^2} = f(t,x).$$

(Here the $G(t-s,x,z)$ is the one previously constructed for the wave equation.) The subscript "ret" stands for *retarded*. It means that the effects of the source

f show up only *later in time*. (Pictorially, a point source at (s, y) emits a wave into the forward-pointing space-time cone of points (t, x) with its vertex at the source. Elsewhere $G_{\mathrm{ret}} = 0$.) Because the wave equation is second-order and time-symmetric, there are infinitely many other Green functions, corresponding to different initial conditions. In particular, there is an *advanced* Green function that absorbs everything and leaves the space empty of waves at later times. For thermodynamic reasons the retarded solution is the relevant one in most applications. (You do not often turn on a flashlight with an incoming wave already focused on it.)

We shall soon see the Duhamel principle at work for Laplace's equation, too.

Coulomb fields

The nonhomogeneous version of Laplace's equation,

$$-\nabla^2 u = j(\mathbf{x}),$$

is called *Poisson's equation.* The corresponding Green function, satisfying

$$-\nabla^2_{(\mathbf{x})} G(\mathbf{x},\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y}),$$

has the physical interpretation of the electrostatic field at $\mathbf{x}$ created by a point charge at $\mathbf{y}$. (The subscript "$(\mathbf{x})$" tells us which variable the operator acts upon.) In dimension 3, with $r \equiv \|\mathbf{x}-\mathbf{y}\|$, it is well known to be

$$G(\mathbf{x},\mathbf{y}) = G(\mathbf{x}-\mathbf{y}) = \frac{1}{4\pi r}$$

(times constants that depend on the system of electrical units being used). In general dimension n (greater than 2, a special case) this becomes

$$G(\mathbf{x},\mathbf{y}) = \frac{C}{r^{n-2}}\,,$$

where $[(n-2)C]^{-1}$ is the "surface area" of the unit $(n-1)$-sphere. For $n = 2$ the formula is

$$G(\mathbf{x},\mathbf{y}) = -\frac{\ln r}{2\pi} = -\frac{\ln r^2}{4\pi}\,.$$

Sketch of proof: For $r \neq 0$ one has in n-dimensional spherical coordinates

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{\partial}{\partial(\text{angles})}\,,$$

so $\nabla^2 r^{2-n} = 0$, as required. Now the hard part is showing that the function has the delta behavior at the origin. Let B_ϵ be the ball of radius ϵ centered at $\mathbf{y}$, and let S_ϵ be its boundary (a sphere of radius ϵ). If we trust that Gauss's theorem continues to hold if delta functions in the derivatives are taken into account, then

$$\int_{B_\epsilon} \nabla^2 G\, d^n z = \int_{S_\epsilon} \hat{\mathbf{n}} \cdot \nabla G\, d^{n-1} S = \int_{S_\epsilon} \frac{\partial G}{\partial r}\, d^{n-1} S$$

$$= C(2-n) \int_{S_\epsilon} \epsilon^{1-n}\, \epsilon^{n-1}\, d(\text{angles})$$

$$= -(n-2)C \times (\text{area of sphere of unit radius}) = -1.$$

Thus the singularity at the origin has the correct normalization. To make a real proof one should do two things: (1) We really need to show, not just that $\int \nabla^2 G = -1$, but that $\int \nabla^2 G(\mathbf{z}) f(\mathbf{z})\, d^n z = -f(\mathbf{0})$ for all smooth functions f. This is not much harder than the calculation just shown: Either use "Green's symmetric identity" (reviewed in a later subsection), or expand f in a power series. All the unwanted terms will go to zero as $\epsilon \to 0$. (2) Strictly speaking, the action of ∇^2 on G is *defined* by integration by parts (in the whole space):

$$\int_{\mathbf{R}^n} \nabla^2 G(\mathbf{z}) f(\mathbf{z})\, d^n z \equiv \int_{\mathbf{R}^n} G(\mathbf{z}) \nabla^2 f(\mathbf{z})\, d^n z,$$

where f is assumed to vanish at infinity. Now apply Gauss's theorem to the *outside* of S_ϵ , where we know it is valid, to show that this integral equals $-f(\mathbf{0})$.

The method of images

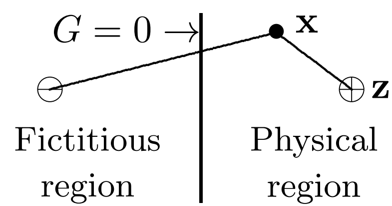
The image method is a generalization of the solution of the wave equation by even and odd periodic extensions, except that this time we extend the Green function instead of the initial data. It is simplest and most intuitive for nonhomogeneous equations, but we'll see that it can easily be extended to homogeneous equations with initial data. It is easy to treat the Poisson and heat equations simultaneously and in parallel.

A single Dirichlet or Neumann boundary

Consider the Poisson Green-function equation

$$-\nabla^2 G(\mathbf{x}, \mathbf{z}) = \delta(\mathbf{x} - \mathbf{z})$$

for $\mathbf{x}$ and $\mathbf{z}$ in a half-space, with the PDE's solutions, and hence G, required to vanish on its boundary (a "perfectly conducting plane" in physics terminology). Start with the Coulomb potential of the source at $\mathbf{z}$ (a positive charge). If we also place a charge of opposite sign in the mirror-image position opposite the charge at $\mathbf{z}$, then *its* Coulomb potential satisfies $\nabla^2 u = 0$ in the physical region (so it doesn't mess up the property $-\nabla^2 G = \delta$), and on the boundary its Coulomb field (gradient) precisely cancels the Coulomb field of the original charge. Success!



To write this construction down algebraically, we need to choose a good coordinate system. Consider $n = 2$ for simplicity; without loss of generality put $\mathbf{z}$ on the x axis, so $\mathbf{z} = (z, 0)$; write $\mathbf{x} = (x, y)$ with the boundary along the y axis, $x = 0$. Then our Green function is

$$G(\mathbf{x}, \mathbf{z}) = -\frac{1}{4\pi} \ln[(x-z)^2 + y^2] + \frac{1}{4\pi} \ln[(x+z)^2 + y^2], \tag{8}$$

because $(x-z)^2+y^2$ is the square of the distance from $\mathbf{x}$ to the positive charge and $(x-z)^2+y^2$ is the square of the distance to the fictitious negative charge. Notice that $G(\mathbf{x}, \mathbf{z}) \neq G(\mathbf{x} - \mathbf{z})$ in this problem, unlike the Coulomb potential and all the other simple Green functions we have seen for *translation-invariant* problems. (This problem is *not* invariant under translations, because the boundary is fixed at $x = 0$.)

The extension of this construction to higher dimensions is easy, but alphabetically inconvenient if you insist on using scalar variables. It would be better to introduce a notation for the components of vectors parallel and perpendicular to the boundary.

Similarly, the Green function for the heat equation on a half-line with $u(0, x) = 0$ is

$$H(t, x, y) - H(t, x, -y) = \frac{1}{\sqrt{4\pi t}}\left[e^{-(x-y)^2/4t} - e^{-(x+y)^2/4t}\right]. \tag{9}$$

This can be shown equal to the Fourier solution

$$\frac{2}{\pi}\int_0^\infty \sin(kx)\sin(ky)e^{-k^2 t}\,dk. \tag{10}$$

Function (9) is the Green function for the nonhomogeneous heat equation with the source at time $s = 0$ (from which the general case can be obtained by the substitution $t \leftarrow t - s$), but by Duhamel's principle it is also the Green function for the homogeneous heat equation with initial data given at $t = 0$, and it is in that role that we have previously encountered (10).

To solve a *Neumann* problem at $x = 0$ ($\frac{\partial u}{\partial x} = 0$ there, so $\frac{\partial G}{\partial x} = 0$), we simply *add* the contribution of the image source instead of subtracting it. This produces a solution that is *even* (rather than odd) under reflection through the boundary plane, and hence its *normal derivative* (rather than the function itself) vanishes on the plane.

The periodic case

Suppose we are interested in the initial-value problem for the heat equation on a ring with coordinate x, $-\pi < x \leq \pi$. We know that the relevant Green function is

$$K(t,x,y) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{in(x-y)} e^{-n^2 t} \tag{11}$$

— from substituting

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-iny} f(y)\, dy \quad \text{into} \quad u(t,x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} e^{-n^2 t}.$$

But another way to get such a Green function is to start from the one for the whole line, H, and add copies of it spaced out periodically:

$$K(t,x,y) = \sum_{M=-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}}\, e^{-(x-y-2\pi M)^2/4t}. \tag{12}$$

Each term of (12) (and hence the whole sum) satisfies the heat equation for $t > 0$. As $t \to 0$ the term with $M = 0$ approaches $\delta(x-y)$ as needed, and all the other terms approach 0 if x and y are in the basic interval $(-\pi, \pi)$. Finally, the function is periodic, $K(t, x + 2\pi, y) = K(t, x, y)$, as desired.

The functions (11) and (12) are equal, although this is not obvious by inspection. Neither sum can be evaluated in closed form in terms of elementary functions. From a numerical point of view they are useful in complementary domains, because the sum in (12) converges very fast when t is small, whereas the one in (11) converges best when t is large.

The equality of (11) and (12) is an instance of the *Poisson summation formula*. This is most easily seen when $x = y$, so that the equality is

$$\sum_{M=-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}}\, e^{-(2\pi M)^2/4t} = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi}\, e^{-n^2 t}. \tag{13}$$

Since

$$H(t,z) = \frac{1}{\sqrt{4\pi t}}\, e^{-z^2/4t} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikz}\, \hat{H}(t,k)\, dk$$

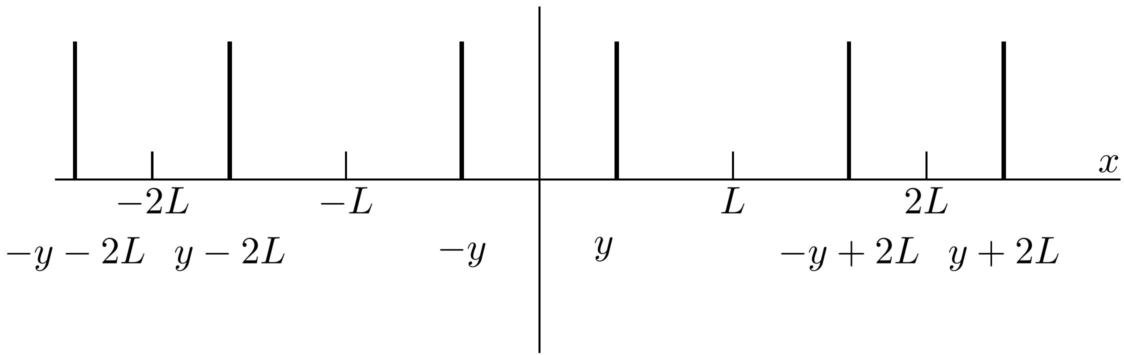
where $\hat{H}(t,k) = \frac{1}{\sqrt{2\pi}}\, e^{-k^2 t}$, (13) is the Poisson relation

$$\sum_{M=-\infty}^{\infty} H(t, 2\pi M) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \hat{H}(t,n).$$

Finite intervals

On the interval $(0, L)$ with Dirichlet boundary conditions at both ends, or Neumann boundary conditions at both ends, we can get the Green function by combining the two previous ideas.

In the Neumann case, reflect the source (at y) through both ends to get "image charges" at $-y$ and $2L-y$. Continue this process indefinitely in both directions to get an infinite sequence of images that build up the needed *even periodic extension* of the delta functions and hence of the Green function and, ultimately, of the solution of the PDE problem.



In the Dirichlet case the first two images are *negative*, and thereafter they alternate in sign so as to build up the odd periodic extensions. (Compare the end of the previous section, where the corresponding linear combination of delta functions was sketched.)

Application of Green's identity

If V is a region in space bounded by a surface S, and u and v are two functions, then Gauss's theorem applied to the vector field $u\nabla v - v\nabla u$ implies

$$\int_S (u\nabla v - v\nabla u) \cdot \hat{\mathbf{n}}\, dS = \int_V (u\nabla^2 v - v\nabla^2 u)\, d^3x. \tag{14}$$

Here $\hat{\mathbf{n}}$ is the outward unit normal vector on S, so $\hat{\mathbf{n}} \cdot \nabla u$ (for example) is the outward *normal derivative* of u, the quantity that appears in Neumann boundary conditions. In the simple regions we have studied so far, it was always possible to write the normal derivative as ($\pm$) the partial derivative in a coordinate that is constant on a portion of the boundary. Formula (14) makes sense and holds true in any dimension, not just 3. It is called *Green's symmetric identity* or *Green's second identity*.

Green's identity has many applications to PDEs, of which we can demonstrate only one of the simplest. Suppose that $G(\mathbf{x},\mathbf{y})$ is the Green function that solves the *homogeneous Dirichlet problem* for the *Poisson equation* in V:

$$-\nabla^2_{(\mathbf{x})} G(\mathbf{x},\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y}) \quad \text{for } \mathbf{x} \in V, \qquad G(\mathbf{x},\mathbf{y}) = 0 \quad \text{for } \mathbf{x} \in S.$$

Let $u(\mathbf{x})$ be any solution of Laplace's equation in V: $\nabla^2 u = 0$. Apply (14) with G in the role of v:

$$\int_V [u(\mathbf{x})\nabla^2 G(\mathbf{x},\mathbf{y}) - G(\mathbf{x},\mathbf{y})\nabla^2 u(\mathbf{x})]\, d^3x = \int_S [u(\mathbf{x})\nabla G(\mathbf{x},\mathbf{y}) - G(\mathbf{x},\mathbf{y})\nabla u(\mathbf{x})]\cdot\hat{\mathbf{n}}\, dS.$$

By the conditions defining G and u, this reduces to

$$u(\mathbf{y}) = \int_V u(\mathbf{x})\delta(\mathbf{x}-\mathbf{y})\, d^3x = -\int_S \hat{\mathbf{n}}\cdot\nabla_{(\mathbf{x})}G(\mathbf{x},\mathbf{y})u(\mathbf{x})\, dS \equiv \int_S g(\mathbf{y},\mathbf{x})u(\mathbf{x})\, dS.$$

This formula expresses u in terms of its Dirichlet data on S. It therefore solves the *nonhomogeneous Dirichlet problem* for *Laplace's equation* in V. This is the version of Duhamel's principle that applies to this situation.

For example, let V be the upper half plane. By the method of images ((8) above with the coordinates turned around), the Green function is

$$G(\mathbf{x},\mathbf{y}) = -\frac{1}{4\pi}\ln[(x_1-y_1)^2+(x_2-y_2)^2] + \frac{1}{4\pi}\ln[(x_1-y_1)^2+(x_2+y_2)^2].$$

(Here $\mathbf{y} = (y_1, y_2)$, etc., and the image charge is at $(y_1, -y_2)$.) To get $g(\mathbf{y},\mathbf{x}) \equiv -\hat{\mathbf{n}}\cdot\nabla_{(\mathbf{x})}G(\mathbf{x}.\mathbf{y})$ we need to differentiate with respect to $-x_2$ (since the outward direction is *down*) and evaluate it at $x_2 = 0$ (the boundary S). This gives

$$-\frac{1}{2\pi}[(x_1-y_1)^2+y_2^2]^{-1}(-y_2)+\frac{1}{2\pi}[(x_1-y_1)^2+y_2^2]^{-1}(+y_2) = \frac{y_2}{\pi}[(x_1-y_1)^2+y_2^2]^{-1}.$$

Reverting to our usual notation ($x_1 \to z$, $y_2 \to y$, $y_1 \to z$) we get

$$g(x, z; y) = \frac{1}{\pi}\frac{y}{(x-z)^2+y^2},$$

our old Green function for this problem!