Polar Coordinates and Bessel Functions

Polar coordinates

The polar coordinates (r, θ) in $\mathbf{R}^2$ are defined by

$$x = r\cos\theta,$$
$$y = r\sin\theta.$$

The usual reason for rewriting a PDE problem in polar coodinates (or another curvilinear coordinate system) is to make the boundary conditions simpler, so that the method of separation of variables can be applied. For example, the vanishing of $u(x, y)$ on a circle is easier to apply when expressed as

$$u(4, \theta) = 0$$

than when stated

$$u(x, y) = 0 \quad \text{whenever } x^2 + y^2 = 16.$$

In fact, the latter can't be satisfied by a nontrivial function of the form $X(x)Y(y)$, as needed by the separation method.

Indeed, a disc of radius r_0 is, in polar coordinates, the region

• disc: $$0 \le r < r_0\,, \quad 0 \le \theta < 2\pi.$$

It is the most obvious of the types of regions that "look like rectangles" when expressed in polar coordinates. Others are

• exterior of disc: $$0 < r_0 < r < \infty, \quad 0 \le \theta < 2\pi;$$

• annulus: $$0 < r_1 < r < r_2\,, \quad 0 \le \theta < 2\pi;$$

• sector: $0 \le r < r_0\,, \quad \Theta_1 \le \theta < \Theta_2\,;$

and three others that have no convenient names (although "partially eaten piece of pie" might do for one of them).

In any such case one will want to rewrite the whole problem in polar coordinates to exploit the geometry. This is likely to make the PDE itself more complicated, however. At least once in your life, you should go through the calculation — using the product rule and multivariable chain rule repeatedly, starting from formulas such as

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta}$$

— that shows that the two-dimensional Laplacian operator

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is equal to

$$\frac{\partial^2}{\partial r^2} + \frac{1}{r}\,\frac{\partial}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2}{\partial \theta^2}\,.$$

It is worth noting that the r-derivative terms

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\,\frac{\partial u}{\partial r}$$

can also be written as a single term,

$$\frac{1}{r}\,\frac{\partial}{\partial r}\left(r\,\frac{\partial u}{\partial r}\right).$$

Let us, therefore, study Laplace's equation

$$0 = \nabla^2 u = \frac{1}{r}\,\frac{\partial}{\partial r}\left(r\,\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\,\frac{\partial^2 u}{\partial \theta^2}\,.$$

We try separation of variables:

$$u(r,\theta) = R(r)\Theta(\theta).$$

We get

$$\frac{1}{r}(rR')'\Theta + \frac{1}{r^2}\,R\Theta'' = 0$$

(where the primes are unambiguous, because each function depends on only one variable). Observe that we can separate the r and θ dependence into different terms by dividing by $R\Theta/r^2$:

$$\frac{r(rR')'}{R} + \frac{\Theta''}{\Theta} = 0.$$

We can therefore introduce an unknown constant (eigenvalue) and split the equation into two ordinary DEs:

$$\frac{\Theta''}{\Theta} = K, \qquad \frac{r(rR')'}{R} = -K.$$

The first of these is our old friend whose solutions are the trig functions; we put it aside to deal with later.

More interesting is the radial equation,

$$(rR')' + \frac{K}{r}R = 0$$

or

$$R'' + \frac{1}{r}\,R' + \frac{K}{r^2}\,R = 0.$$

It is of the general Sturm–Liouville type. Consulting the theorems and definitions concerning those, we see that we will have a *regular* Sturm–Liouvile problem provided that the boundaries of our region are

$$r = r_1 \quad \text{and} \qquad r = r_2 \qquad \text{with } r_1 > 0 \text{ and } r_2 < \infty$$

— that is, for the half-eaten piece of pie and the annulus (ring). For the more common situations of the disc, disc exterior, and sector, the SL problem is *singular.*

However, a little learning is a dangerous thing. Although the analysis I have just given you is correct, and will be valuable soon when we complicate the equation by adding another term, it turns out to be unnecessary in the present case. Let's make the change of variables

$$z \equiv \ln r \qquad (\text{hence } r = e^z),$$

so that

$$\frac{d}{dr} = \frac{dz}{dr}\,\frac{d}{dz} = \frac{1}{r}\,\frac{d}{dz}\,.$$

Then

$$rR' = \frac{R}{z}, \qquad (rR')' = \frac{1}{r}\,\frac{d^2R}{dz^2}\,,$$

so the equation becomes

$$\frac{d^2R}{dz^2} + KR = 0.$$

It is our old friend after all!

Let us record its (basis) solutions for the various classes of K:

1. $K = -\lambda^2 < 0:$ $\qquad R = e^{\pm\lambda z} = r^{\pm\lambda}.$

2. $K = 0:$ $\qquad R = 1 \quad \text{and} \quad R = z = \ln r.$

3. $K = \mu^2 > 0:$ $\qquad R = e^{\pm i\mu z} = r^{\pm i\mu};$

   that is, $R = \cos(\mu \ln r)$ and $R = \sin(\mu \ln r)$.

Boundary conditions in polar coordinates

We consider two examples and make brief remarks on a third.

I. Interior of a disc of radius r_0

Three boundary conditions, of different natures, arise here.

First, since the coordinate θ goes "all the way around", $u(r,\theta)$ must be periodic in θ with period 2π. Therefore, the solutions of the angular equation, $\Theta'' = K\Theta$, will be the terms of a full Fourier series at the standard scale:

$$u(r,\theta) = \sum_{n=-\infty}^{\infty} c_n \, e^{in\theta} \, R_n(r).$$

(Of course, we could use sines and cosines instead.) Moreover, $K = -n^2$.

Second, at the rim of the disc a well-posed potential problem requires a standard nonhomogeneous Dirichlet, Neumann, or Robin condition, such as

$$u(r_0,\theta) = f(\theta).$$

This will be applied to the whole series, not each term R_n, and will eventually determine the coefficients c_n.

Third, to complete the specification of R_n we need to say how the solution behaves as $r \to 0$. We know that $u(r, \theta)$ reexpressed as a function of x and y must be a *solution* at the center of the disc. This implies that $R_n(r)$ must stay bounded as r approaches 0. Looking back at our list of possible radial solutions, we see that the allowable ones are $R_0(r) = 1$ and $R_n(r) = r^{|n|}$ for $n \neq 0$.

So, finally, the solution is

$$u(r, \theta) = \sum_{n=-\infty}^{\infty} c_n \, e^{in\theta} \, r^{|n|},$$

where c_n must be determined by (in the Dirichlet case)

$$f(\theta) = \sum_{n=-\infty}^{\infty} c_n \, e^{in\theta} \, r_0^{|n|};$$

that is,

$$c_n = \frac{1}{r_0^{|n|}} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} f(\theta)\, d\theta.$$

II. A partially eaten piece of pie

Consider the truncated sector, or polar rectangle, bounded by the four curves

$$r = r_1\,, \quad r = r_2\,, \quad \theta = \theta_1\,, \quad \theta = \theta_2\,,$$

where $0 < r_1$ and $r_2 < \infty$. In this case, all four boundaries are of the "regular" type. Let's suppose that nonhomogeneous data are given on all four sides — something like

$$u(r_1,\theta) = f_1(\theta), \quad u(r_2,\theta) = f_2(\theta),$$

$$\frac{\partial u}{\partial \theta}(r,\theta_1) = f_3(r), \quad \frac{\partial u}{\partial \theta}(r,\theta_2) = f_4(r).$$

As in the Cartesian rectangle case, before separating variables we must split this into two problems, one with homogeneous θ boundary conditions and one with homogeneous r boundary conditions. Let us say $u = v + w$, where v and w individually solve the potential equation, v satisfies

$$v(r_1, \theta) = 0, \quad v(r_2, \theta) = 0, \quad \frac{\partial v}{\partial \theta}(r, \theta_1) = f_3(r), \quad \frac{\partial v}{\partial \theta}(r, \theta_2) = f_4(r),$$

and w satisfies

$$w(r_1, \theta) = f_1(\theta), \quad w(r_2, \theta) = f_2(\theta), \quad \frac{\partial w}{\partial \theta}(r, \theta_1) = 0, \quad \frac{\partial w}{\partial \theta}(r, \theta_2) = 0.$$

In solving for v, it is the homogeneous conditions on R that must determine the allowed eigenvalues K. Thus here, for the first time, we really treat the radial equation as a Sturm–Liouville problem. In order for R to vanish at both r_1 and

r_2, we must have $K > 0$, the third case in our list of radial solutions. That is, for each normal mode we have an eigenvalue $K_\mu = \mu^2$ and an eigenfunction

$$R_\mu(r) = A_\mu \cos(\mu \ln r) + B_\mu \sin(\mu \ln r)$$

(or, alternatively, $R_\mu = C_{\mu,+} r^{i\mu} + C_{\mu,-} r^{-i\mu}$). The two equations

$$R(r_1) = 0 = R(r_2)$$

(1) determine a discrete list of allowable values of μ, and (2) determine the ratio of A_μ to B_μ (or $C_{\mu,+}$ to $C_{\mu,-}$). This leaves an overall constant factor in R_μ undetermined, as is always the case in finding normal modes. I postpone the details of this calculation for a moment; the principle is the same as in the very first separation-of-variables problem we did, where the eigenvalues turned out to be $(n\pi/L)^2$ and the eigenfunctions $\sin(n\pi x/L)$ times an arbitrary constant.

To finish the solution for v we need to solve the equation

$$\Theta'' = +\mu^2 \Theta.$$

Thus the angular dependence of this solution is exponential, not trigonometric. We can write

$$v(r,\theta) = \sum_\mu c_\mu \, R_\mu(r) \, \left(C_\mu e^{\mu\theta} + D_\mu e^{-\mu\theta} \right).$$

The constants C and D are to be determined by imposing the remaining boundary conditions,

$$\frac{\partial v}{\partial \theta}(r,\theta_1) = f_3(r), \qquad \frac{\partial v}{\partial \theta}(r,\theta_2) = f_4(r).$$

In general this will be a coupled pair of Sturm–Liouville expansions in the orthogonal eigenfunctions $R_\mu(r)$.

That's v; now we need to find w. That problem is like this one, except that the roles of r and θ are interchanged. The result will be a Fourier cosine series in θ with radial factors that depend exponentially on $\ln r$; that is, linear combinations of r^n and r^{-n}. I hope that by now I can leave the details to your imagination.

Unfinished business: Let us consider the details of finding the eigenvalues μ and eigenfunctions R_μ. The two relevant algebraic equations are

$$0 = R_\mu(r_1) = C_{\mu,+} r_1^{i\mu} + C_{\mu,-} r_1^{-i\mu}$$

and

$$0 = R_\mu(r_2) = C_{\mu,+} r_2^{i\mu} + C_{\mu,-} r_2^{-i\mu}.$$

A nontrivial solution will exist if and only if the determinant vanishes:

$$0 = \begin{vmatrix} r_1^{i\mu} & r_1^{-i\mu} \\ r_2^{i\mu} & r_2^{-i\mu} \end{vmatrix} = \left(\frac{r_1}{r_2}\right)^{i\mu} - \left(\frac{r_1}{r_2}\right)^{-i\mu}.$$

This is proportional to

$$\sin\bigl(\mu \ln(r_1/r_2)\bigr),$$

so it vanishes precisely when μ is an integer multiple of the constant $\pi/\ln(r_1/r_2)$.

Returning to one of the linear algebraic equations, we find

$$\frac{C_{\mu,+}}{C_{\mu,-}} = -r_1^{-2i\mu} = -e^{-2i\mu \ln r_1}.$$

(Using the other equation would give $C_{\mu,+}/C_{\mu,-} = -e^{-2i\mu \ln r_2}$, but these two equations are equivalent because of the eigenvalue condition, which may be rewritten as $\mu \ln r_1 - \mu \ln r_2 = N\pi$.) The neatest (albeit not the most obvious) normalization convention is to choose $C_{\mu,-} = -r_1^{i\mu}$; then $C_{\mu,+} = r_1^{-i\mu}$, and

$$\begin{aligned} R_\mu(r) &\equiv C_{\mu,+} r^{i\mu} + C_{\mu,-} r^{-i\mu} \\ &= \left(\frac{r}{r_1}\right)^{i\mu} - \left(\frac{r}{r_1}\right)^{-i\mu} \\ &= 2i \sin\bigl(\mu \ln(r/r_1)\bigr). \end{aligned}$$

Thus the Sturm–Liouville expansion involved in this problem is simply an ordinary Fourier sine series, though expressed in very awkward notation because of the context in which it arose. (In our usual notation, we have $L \equiv \ln(r_1/r_2)$, $\mu = n\pi/L$, $x = z - \ln r_1$, $C_{\mu,+} = b_n/2i$.) We would have encountered the same complications in Cartesian coordinates if we had considered examples where none of the boundaries lay on the coordinate axes (but the boundaries were parallel to the axes).

III. A sector (the pie intact)

Consider the region

$$0 \leq r < r_2 \, , \quad \Theta_1 \leq \theta < \Theta_2 \, ,$$

with nonhomogeneous data on the straight sides. (This is the limiting case of the v problem above as $r_1 \to 0$.) The endpoint $r = 0$ is singular, so we are not guaranteed that a standard Sturm–Liouville expansion will apply. Indeed, in terms of the variable $z = \ln r$, where the radial equation becomes trivial, the endpoint is at $z = -\infty$. This problem is therefore a precise polar analogue of the infinite rectangular slot problem, and the solution will be a Fourier sine or cosine *transform* in a variable $\zeta \equiv -z + C$ that vanishes when $r = r_2$. (That is, $C = \ln r_2$.)