Moving into Higher Dimensions: The Rectangle

We will now work out a big example problem. It will break up into many small examples, which will demonstrate many of the principles we've talked about — often in a slightly new context.

Problem statement

We will consider heat conduction in a two-dimensional region, a rectangle. The ranges of the variables, therefore, will be

$$0 < x < a, \qquad 0 < y < b, \qquad t > 0.$$

Without loss of generality, we can assume that the variables have been scaled so that $a = \pi$.

The heat equation is

$$\text{PDE:} \qquad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\,.$$

Let us assume that the boundary conditions are

$$\text{BC}_1: \qquad \frac{\partial u}{\partial x}(t,0,y) = 0 = \frac{\partial u}{\partial x}(t,\pi,y)\,,$$

$$\text{BC}_2: \qquad u(t,x,0) = p(x), \qquad u(t,x,b) = q(x).$$

That is, the plate is insulated on the sides, and the temperature on the top and bottom edges is known and given by the functions p and q. Finally, there will be some initial temperature distribution

$$\text{IC:} \qquad u(0,x,y) = f(x,y).$$

From our experience with the one-dimensional problem, we know that we must eliminate the nonhomogeneous boundary condition (BC_2) before we can solve the initial-value problem by separation of variables! Fortunately, p and q are independent of t, so we can do this by the same technique used in one dimension: hunt for a time-independent solution of (PDE) and (BC), $v(t,x,y) = V(x,y)$, then consider the initial-value problem with homogeneous boundary conditions satisfied by $u - v$.

So, we first want to solve

PDE: $$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0,$$

BC_1: $$\frac{\partial V}{\partial x}(0,y) = 0 = \frac{\partial V}{\partial x}(\pi,y)\,,$$

BC$_2$: $$V(x,0) = p(x), \qquad V(x,b) = q(x).$$

This is still a *partial* differential equation (namely, the two-dimensional *Laplace equation*). Furthermore, it still contains two nonhomogeneous conditions. Therefore, we split the problem again:

$$V = V_1 + V_2\,,$$

$$\begin{aligned} V_1(x,0) &= p(x), & V_2(x,0) &= 0, \\ V_1(x,b) &= 0, & V_2(x,b) &= q(x). \end{aligned}$$

Each V_j is supposed to satisfy Laplace's equation and (BC$_1$).

Remark: This splitting is slightly different from the one involving the steady-state solution. In each subproblem here we have replaced every nonhomogeneous condition except one by its *corresponding homogeneous condition*. In contrast, for the steady-state solution we simply *discarded* the inconvenient nonhomogeneous condition, and later will modify the corresponding nonhomogeneous condition in the other subproblem to account for the failure of the steady-state solution to vanish on that boundary. Which of these techniques is best varies with the problem, but the basic principle is the same: Work with only one nonhomogeneous condition at a time, so that you can exploit the superposition principle correctly.

Let us solve for V_2 by separation of variables:

$$V_{2\text{sep}}(x, y) = X(x)Y(y).$$

$$0 = X''Y + XY'' \;\Rightarrow\; -\frac{X''}{X} = \lambda = \frac{Y''}{Y}\,.$$

The boundary condition (BC_1) implies that

$$X'(0) = 0 = X'(\pi).$$

Therefore, up to a constant,

$$X(x) = \cos nx, \qquad \lambda = n^2.$$

Now Y must be a solution of $Y'' = n^2 Y$ that vanishes at $y = 0$; that is, up to a constant,

$$Y(y) = \sinh ny \quad \text{if } n \neq 0.$$

The case 0 must be treated separately: $Y(y) = y$. We have now taken care of three of the four boundaries. The remaining boundary condition is nonhomogeneous, and thus we cannot apply it to the individual separated solutions XY; first we must adding up the separated solutions with arbitrary coefficients:

$$V_2(x, y) = a_0 y + \sum_{n=1}^{\infty} a_n \cos nx \sinh ny.$$

Now we must have

$$q(x) = a_0 b + \sum_{n=0}^{\infty} a_n \cos nx \sinh nb.$$

This is a Fourier cosine series, so we solve for the coefficients by the usual formula:

$$a_n \sinh nb = \frac{2}{\pi} \int_0^\pi \cos nx \, q(x) \, dx \qquad (n > 0).$$

Divide by $\sinh nb$ to get a formula for a_n. For $n = 0$ the Fourier formula lacks the factor 2, and we end up with

$$a_0 = \frac{1}{\pi b} \int_0^\pi q(x) \, dx.$$

This completes the solution for V_2.

Solving for V_1 is exactly the same except that we need $Y(b) = 0$ instead of $Y(0) = 0$. The appropriate solution of $Y'' = n^2 Y$ can be written as a linear combination of $\sinh ny$ and $\cosh ny$, or of e^{ny} and e^{-ny}, but it is neater to write it as

$$Y(y) = \sinh\bigl(n(y - b)\bigr),$$

which manifestly satisfies the initial condition at b as well as the ODE. (Recall that hyperbolic functions satisfy trig-like identities, in this case

$$\begin{aligned}\sinh\bigl(n(y-b)\bigr) &= \cosh nb \sinh ny - \sinh nb \cosh ny \\ &= \tfrac{1}{2}e^{-nb}\, e^{ny} - \tfrac{1}{2}e^{nb}\, e^{-ny},\end{aligned}$$

so the three forms are consistent.) Again the case $n = 0$ is special: $Y(y) = y - b$. We now have

$$V_1(x, y) = A_0(y - b) + \sum_{n=1}^{\infty} A_n \cos nx \sinh n(y - b).$$

At $y = 0$ this becomes

$$p(x) = -A_0 b - \sum_{n=1}^{\infty} A_n \cos nx \sinh nb.$$

Thus

$$A_n = -\frac{2}{\pi \sinh nb} \int_0^\pi \cos nx \, p(x) \, dx \qquad (n > 0),$$

$$A_0 = -\frac{1}{\pi b} \int_0^\pi p(x) \, dx.$$

This completes the solution for V_1 and hence for $v(t,x,y)$.

Remark: Since the boundary conditions at $y = 0$ and $y = b$ refer to the *same variable*, it was not really necessary to treat them separately. We could have separated variables in the problem [(Laplace PDE) + (BC_1)] satisfied by the function V, getting

$$V_{\text{sep}}(x,y) = \cos nx \, Y(y), \qquad Y'' = n^2 Y.$$

Then we could find the *general solution* of this last equation,

$$Y(y) = a_n \sinh ny + b_n \cosh ny$$

— or, better,

$$Y(y) = a_n \sinh ny + A_n \sinh n(y-b);$$

write the general superposition as a sum of these over n; and then use the two nonhomogeneous boundary conditions (BC_2) to determine the constants a_n and A_n in the summation.

This works because the nonhomogeneous conditions refer to parallel parts of the boundary. It definitely will not work for perpendicular edges! When in doubt, follow the injunction to deal with just one nonhomogeneity at a time.

Homogeneous problem

Next we're supposed to solve for $w \equiv u - v$, which must satisfy

PDE: $$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\,,$$

BC$_1$: $$\frac{\partial w}{\partial x}(t,0,y) = 0 = \frac{\partial w}{\partial x}(t,\pi,y)\,,$$

BC$_2$: $$w(t,x,0) = 0, \qquad w(t,x,b) = 0,$$

IC: $$w(0,x,y) = f(x,y) - V(x,y) \equiv g(x,y).$$

Since there is only one nonhomogeneous condition, we can separate variables immediately:

$$w_{\text{sep}}(t,x,y) = T(t)X(x)Y(y).$$

$$T'XY = TX''Y + TXY''.$$

$$\frac{T'}{T} = \frac{X''}{X} + \frac{Y''}{Y} = -\lambda.$$

(We know that λ is a constant, because the left side of the equation depends only on t and the right side does not depend on t at all. By analogy with the one-dimensional case we can predict that λ will be positive.) Since X''/X depends only on x and Y''/Y depends only on y, we can introduce *another* separation constant:

$$\frac{X''}{X} = -\mu, \qquad \frac{Y''}{Y} = -\lambda + \mu.$$

The boundary conditions translate to

$$X'(0) = 0 = X'(\pi), \qquad Y(0) = 0 = Y(b).$$

Thus for X we have the familiar solution

$$X(x) = \cos mx, \qquad \mu = m^2.$$

Similarly, we must have

$$Y(y) = \sin \frac{n\pi y}{b}, \qquad -\lambda + \mu = -\frac{n^2\pi^2}{b^2}$$

$$\Rightarrow \lambda = m^2 + \frac{n^2\pi^2}{b^2} \equiv \lambda_{mn} \,.$$

Then

$$T(t) = e^{-\lambda t}.$$

(As usual in separation of variables, we have left out all the arbitrary constants multiplying these solutions. They will all be absorbed into the coefficients in the final Fourier series.)

We are now ready to superpose solutions and match the initial data. The most general solution of the homogeneous problem is a double infinite series,

$$w(t,x,y) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} c_{mn} \cos mx \, \sin \frac{n\pi y}{b} \, e^{-\lambda_{mn} t}.$$

The initial condition is

$$g(x,y) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} c_{mn} \cos mx \, \sin \frac{n\pi y}{b} \,. \qquad (*)$$

To solve for c_{mn} we have to apply Fourier formulas twice:

$$\sum_{m=0}^{\infty} c_{mn} \cos mx = \frac{2}{b} \int_0^b \sin \frac{n\pi y}{b} \, g(x, y) \, dy;$$

$$c_{mn} = \frac{2}{\pi} \frac{2}{b} \int_0^{\pi} dx \int_0^b dy \, \cos mx \, \sin \frac{n\pi y}{b} \, g(x, y) \qquad (m > 0),$$

$$c_{0n} = \frac{2}{\pi b} \int_0^{\pi} dx \int_0^b dy \, \sin \frac{n\pi y}{b} \, g(x, y).$$

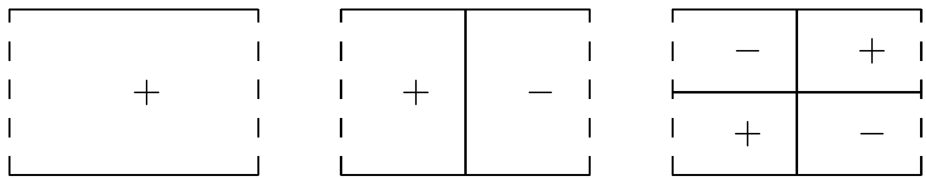
This completes the solution for w. Now we have the full solution to the original problem:

$$u(t, x, y) = w(t, x, y) + V(x, y).$$

Furthermore, along the way we have constructed a very interesting family of functions defined on the rectangle:

$$\phi_{mn}(x,y) \equiv \cos mx \sin \frac{n\pi y}{b}\,.$$

A few early members of the family look like this:



$\sin \frac{\pi y}{b}$ $\qquad$ $\cos x \sin \frac{\pi y}{b}$ $\qquad$ $\cos x \sin \frac{2\pi y}{b}$

(Recall that $\cos(0x) = 1$.) The function is positive or negative in each region according to the sign shown. The function is zero on the solid lines and its normal derivative is zero along the dashed boundaries. The functions have these key properties for our purpose:

- They are *eigenvectors* of the Laplacian operator:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \phi_{mn} = -\lambda_{mn}\phi_{mn} \, .$$

- *Completeness:* Any function (reasonably well-behaved) can be expanded as an infinite linear combination of them (the double Fourier series $(*)$).

- *Orthogonality:* Each expansion coefficient c_{mn} can be calculated by a relatively simple integral formula, involving the corresponding eigenfunction ϕ_{mn} only.

These functions form an *orthogonal basis* for the vector space of functions whose domain is the rectangle (more precisely, for the space L^2 of square-integrable functions on the rectangle), precisely analogous to the orthogonal basis of eigenvectors for a symmetric matrix that students learn to construct in linear-algebra or ODE courses.

Remark: A complete treatment of convergence issues for the double Fourier series is not feasible here. We can say that if $g(x,y)$ is very smooth, then the coefficients go to 0 fast as m or $n \to \infty$, and everything is OK. (More precisely, what needs to be smooth is the extension of g which is even and periodic in x and odd periodic in y. This places additional conditions on the behavior of g at the boundaries.) Also, if g is merely square-integrable, then the series converges in the mean, but not necessarily pointwise. (In that case the series for g can be used for certain theoretical purposes — e.g., inside the integrand of certain integrals — but an attempt to add it up on a computer is likely to lead to disaster.) However, when $t > 0$ the series for w will converge nicely, even if g is rough, because the exponential factors make the terms decrease rapidly with m and n. This is a special feature of the heat equation: Because it describes a diffusive process, it drastically smooths out whatever initial data is fed into it.

The consistency condition in the Neumann problem

Go back now to the steady-state problem and suppose that the boundary conditions on all four sides of the rectangle are of the normal-derivative type:

PDE: $$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0,$$

BC$_1$: $$\frac{\partial V}{\partial x}(0, y) = f(y), \qquad \frac{\partial V}{\partial x}(\pi, y) = g(y),$$

BC$_2$: $$\frac{\partial V}{\partial y}(x, 0) = p(x), \qquad \frac{\partial V}{\partial y}(x, b) = q(x).$$

Apply the two-dimensional version of Gauss's theorem:

$$
\begin{aligned}
0 &= \int_0^\pi dx \int_0^b dy\, \nabla^2 V \\
&= \int_C \hat{\mathbf{n}} \cdot \nabla V \, ds \\
&= -\int_0^\pi f(y)\, dy + \int_0^\pi g(y)\, dy - \int_0^b p(x)\, dx + \int_0^b q(x)\, dx.
\end{aligned}
$$

Without even attempting to solve the problem, we can see that *there is no solution unless the net integral of the (outward) normal derivative data around the entire perimeter of the region is exactly equal to zero.*

This fact is easy to understand physically if we recall that this problem arose from a time-dependent problem of heat conduction, and that a Neumann boundary condition is a statement about *heat flow* out of the region concerned.

If there is a net heat flow out of the region (and no heat source in the interior), then the rectangular object ought to be cooling off! It is not surprising that no *steady-state* solution can exist.

This existence problem is accompanied by a phenomenon of nonuniqueness, as often happens with linear equations. (Remember what happens to N equations in N unknowns when the determinant of the coefficient matrix is 0.) Suppose that the net heat flux *is* zero, and that we have found a solution, V, of the steady-state problem. Add a constant: $V_*(x) \equiv V(x) + C$. Since the constant function has zero Laplacian *and* zero normal derivatives all around, V_* is also a solution, no matter what C is. In the context of the original time-dependent heat problem, this ambiguity in the definition of the steady-state solution is merely a harmless nuisance: Just subtract C from the initial data ($g(x, y)$) of the complementary problem with homogeneous boundary data, and the final solution will come out the same (unique).

The common lesson of these two examples is, "Just because you *can* expand an unknown solution in a Fourier series doesn't mean that you *should*." Sometimes a simply polynomial will do a better job.

Retaining consistency in the Neumann problem

Consider Laplace's equation in a rectangle with Neumann boundary conditions as above, and assume that the normal derivatives integrate to 0, so a solution should exist. Let's reform the notation to make it more systematic:

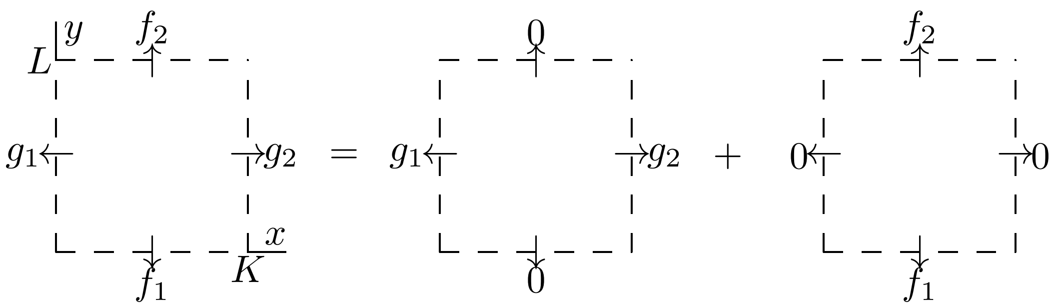
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

$$-\frac{\partial u}{\partial y}(x,0) = f_1(x), \qquad \frac{\partial u}{\partial y}(x,L) = f_2(x),$$

$$-\frac{\partial u}{\partial x}(0,y) = g_1(y), \qquad \frac{\partial u}{\partial x}(K,y) = g_2(y),$$

with

$$\int_0^K [f_1(x) + f_2(x)]\,dx + \int_0^L [g_1(y) + g_2(y)]\,dy = 0.$$



Following the usual strategy, let's break up the problem into two, so that we have nonhomogeneous data in only one variable at a time. (The diagram indicates the resulting boundary equations.) But we have outfoxed ourselves.* There is no reason why $\int_0^K [f_1(x)+f_2(x)]\,dx$ and $\int_0^L [g_1(y)+g_2(y)]\,dy$ should equal 0 individually, so in general the two subproblems will not have solutions. What to do?

Here is a "magic rescue". The function $V(x,y) \equiv x^2 - y^2$ satisfies $\nabla^2 V = 0$ and

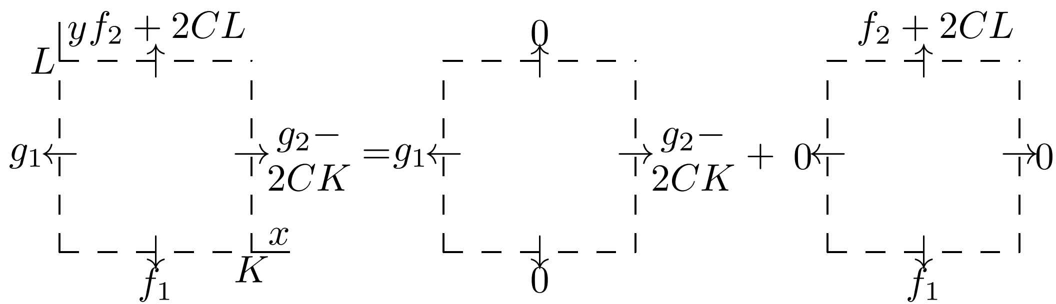
$$\frac{\partial V}{\partial x} = 2x = \begin{cases} 0 & \text{when } x = 0, \\ 2K & \text{when } x = K, \end{cases} \qquad \frac{\partial V}{\partial y} = -2y = \begin{cases} 0 & \text{when } y = 0, \\ -2L & \text{when } y = L. \end{cases}$$

Let

$$C = -\frac{1}{2KL}\int_0^K [f_1(x) + f_2(x)]\,dx = +\frac{1}{2KL}\int_0^L [g_1(y) + g_2(y)]\,dy.$$

* Pointed out by Juan Carcuz-Jerez, a student in Fall 2000 class.

We would like to have a solution, $u(x, y)$, of the original problem with data f_1, f_2, g_1, g_2. Suppose for a moment that such a solution exists, and consider $w \equiv u - CV$. We see that $\nabla^2 w = 0$ and that w satisfies Neumann boundary conditions shown in the next diagram, along with the obvious decomposition:



We calculate

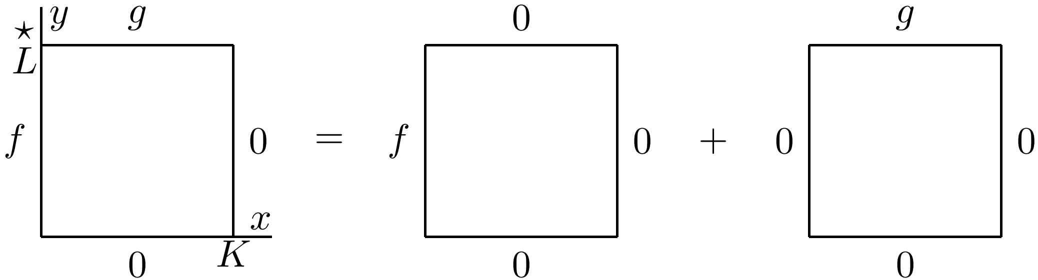
$$\int_0^L [g_1(y) + g_2(y) - 2CK]\, dy = 2CKL - 2CKL = 0,$$
$$\int_0^K [f_1(x) + f_2(x) + 2CK]\, dx = -2CKL + 2CKL = 0.$$

Therefore, each of these subproblems *does* have a solution, which can be constructed as a Fourier cosine series in the usual way. (As usual in pure Neumann problems, the solutions are nonunique because an arbitrary constant could be added. Apart from that, the $n = 0$ term in each cosine series is a function that is independent of the Fourier variable and linear in the other variable. (Try it and see!))

We can now define $u = w + CV$ and observe that it solves the original Laplacian problem. (Hence it could serve as the steady-state solution for a related heat or wave problem.)

Avoiding poor convergence at corners

Consider a Dirichlet problem for Laplace's equation with two nonhomogeneous conditions:



The two subproblems are solved by Fourier sine series in the usual way. Unless $f(0) = 0 = f(L)$ and $g(0) = 0 = g(K)$, the solutions will demonstrate nonuniform convergence (and the Gibbs phenomenon). Suppose, however, that f and g are continuous (and piecewise smooth) and

$$f(0) = 0, \qquad g(K) = 0, \qquad f(L) = g(0) \neq 0.$$

Then the boundary data function is *continuous* all around the boundary, and one suspects that the optimal Fourier solution should be better behaved. The standard decomposition has introduced an *artificial discontinuity* at the corner marked "$\star$" and thus a *spurious difficulty* of poor convergence.

The cure for this (admittedly relatively mild) disease is to consider

$$V(x,y) \equiv -g(0)\,\frac{y}{L}\,\frac{x-K}{K}\,.$$

We see that $\nabla^2 V = 0$ and

$$V(\star) \equiv V(0,L) = g(0), \qquad V(0,0) = V(K,0) = V(K,L) = 0.$$

Therefore, $w \equiv u - V$ satisfies $\nabla^2 w = 0$ with Dirichlet boundary data that *vanish at all four corners*. The problem for w can be decomposed into two subproblems in the usual way, and both of those will have uniformly convergent Fourier sine series.

More generally, any function of the form

$$V(x,y) = A + Bx + Cy + Dxy$$

is a solution of Laplace's equation. Given any continuous boundary data around a rectangle, the constants $A, B, C.D$ can be chosen so that V matches the data exactly at all four corners. Then $W \equiv u - V$ has continuous data that vanish at all four corners. By prudently subtracting off V before separating variables we get a better behaved Fourier solution. Of course, the double Fourier sine series for $V(x,y)$ itself *would* exhibit nonuniform convergence, but there is no need here to decompose the simple polynomial function V in that way.)