Test A – Solutions

Calculators may be used for simple arithmetic operations only!

When a question appears in two versions, answer the version appropriate to your status (honors or regular). Then work on the other version if you have time.

1. (15 pts.) Classify each of these equations as linear homogeneous, linear nonhomogeneous, or nonlinear. (In each case, $u$ is the unknown function.)

   (a) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 3$.  linear nonhomogeneous

   (b) $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + x^2 u$.  linear homogeneous

   (c) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = \cos u$.  nonlinear

2. (40 pts.)

   (a) Find the Fourier cosine series for the function defined by

   \[ f(x) = x - \pi \] on the interval $0 \leq x \leq \pi$.

   (Write down the form of the series, then write down the integrals for the coefficients. Don’t evaluate the integrals, just take my word for it that their values decrease as $n^{-2}$.)

   \[ x - \pi \sim \sum_{n=0}^{\infty} a_n \cos(nx). \]

   \[ a_0 = \frac{1}{\pi} \int_{0}^{\pi} (x - \pi) \, dx, \]

   whereas for $n > 0$ we have

   \[ a_n = \frac{2}{\pi} \int_{0}^{\pi} (x - \pi) \cos(nx) \, dx. \]

   (b) Your series represents a periodic function defined on the whole line $-\infty < x < \infty$. Sketch the graph of that function. (Let your $x$ axis run from $-3\pi$ to $3\pi$.)

   The function is even under reflection through each endpoint.
(c) (honors)

(1) Does the series converge pointwise?
Yes. \( f \) is piecewise smooth.

(2) Does it converge uniformly?
Yes. (The periodically extended function is continuous as well as piecewise smooth. Alternatively, \( a_n \sim n^{-2} \) is sufficient to guarantee that the series converges uniformly.)

(3) Does it converge in the mean?
Yes. \( f \) is square-integrable.

(c) (regular) Use the series in (a) to solve the heat-conduction problem

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad (0 < x < \pi, \: -\infty < t < \infty),
\]

\[
\frac{\partial u}{\partial x}(0, t) = 0, \quad \frac{\partial u}{\partial x}(\pi, t) = 0 \quad (-\infty < t < \infty),
\]

\[
u(x, 0) = f(x) \quad (0 < x < \pi).
\]

(Recall that separation of variables in the heat equation gives the equations

\[X''(x) = -n^2 X(x), \quad T'(t) = -n^2 T(t), \quad X'(0) = 0 = X'(\pi).\]

Don’t stop to rederive this.)

\[u(x, t) = \sum_{n=0}^{\infty} a_n \cos(nx) e^{-n^2 t}.
\]

3. (37 pts.) Consider the wave propagation problem

\[
\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad (0 < x < \infty, \: -\infty < t < \infty),
\]

\[
\frac{\partial u}{\partial x}(0, t) = 0 \quad (-\infty < t < \infty),
\]

with initial data

\[u(x, 0) = f(x), \quad \frac{\partial u}{\partial t}(x, 0) = g(x) \quad (0 < x < \infty).
\]

(a) Write down the d’Alembert (also called “method of characteristics”) formula for the solution, making sure everything is defined.

Extend \( f \) and \( g \) so that they are even (under reflection through the origin). (I use the same letters, \( f \) and \( g \), for both the original data functions and their extensions.) Let \( G(x) \) be the antiderivative of \( g \) with \( G(0) = 0 \). (Then \( G(x) \) is an odd function, and \( f'(x) \) is also odd.) Now the solution is

\[u(x, t) = \frac{1}{2} [f(x + t) + f(x - t)] + \frac{1}{2} [G(x + t) - G(x - t)].\]

Alternatively, the \( G \) terms can be written as

\[\frac{1}{2} \int_{x-t}^{x+t} g(w) \, dw.
\]
(b) Let \( h(x) \) be a nonnegative function sharply peaked around \( x = 1 \) (for example, \( h(x) = e^{-10(x-1)^2} \)). Sketch the solution, \( u(x,t) \), as a function of \( x \) for \( t = 0, t = 1, t = 2, \) and \( t = 3 \), for the initial data

\[
(\text{regular}) \quad u(x,0) = h(x), \quad \frac{\partial u}{\partial t}(x,0) = 0.
\]

The initial blip divides into left-moving and right-moving pulses. With this boundary condition the left-moving pulse does not invert when it reflects; at \( t = 1 \) the “ghost” pulse from negative \( x \) restores it to its original height (but only half its original width).

\[
\begin{align*}
 t = 3 & \\
 t = 2 & \\
 t = 1 & \\
 t = 0 &
\end{align*}
\]

\[
\begin{align*}
 u, t & \\
 \bigtriangleup & \\
 \bigtriangleup & \\
 \bigtriangleup & \\
 \bigtriangleup & \\
 \bigtriangleup & \\
 x, t &
\end{align*}
\]

\[
(\text{honors}) \quad u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = h(x).
\]

I’ll do this carefully in \textit{Mathematica} and post it in a separate file (along with some intermediate times).

4. \( (8 \text{ pts.}) \) Use the complex exponential function to prove that

\[
\sin(x + y) = \sin x \cos y + \cos x \sin y.
\]

The basic definition is

\[
e^{\pm ix} = \cos x \pm i \sin x.
\]

Therefore,

\[
\cos x = \frac{1}{2}(e^{ix} + e^{-ix}), \quad \sin x = \frac{1}{2i}(e^{ix} - e^{-ix}).
\]

Substitute these into the right side of the identity:

\[
\sin x \cos y + \cos x \sin y = \frac{1}{4i}[e^{ix} - e^{-ix})(e^{iy} + e^{-iy}) + (e^{ix} + e^{-ix})(e^{iy} - e^{-iy})],
\]

which simplifies to

\[
\frac{1}{2i}(e^{i(x+y)} - e^{-i(x+y)}) = \sin(x + y),
\]

as expected.