Math. 412 (Fulling) 25 September 2015

Test A – Solutions

Calculators may be used for simple arithmetic operations only!

When a question appears in two versions, answer the version appropriate to your status (honors or regular). Then work on the other version if you have time.

1. *(15 pts.)* Classify each of these equations as linear homogeneous, linear nonhomogeneous, or nonlinear. (In each case, u is the unknown function.)

(a) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 3\,.$ linear nonhomogeneous

(b) $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + x^2 u\,.$ linear homogeneous

(c) $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial t} = \cos u\,.$ nonlinear

2. *(40 pts.)*

(a) Find the Fourier cosine series for the function defined by

$$f(x) = x - \pi \quad \text{on the interval } 0 \le x \le \pi\,.$$

(Write down the form of the series, then write down the integrals for the coefficients. *Don't* evaluate the integrals, just take my word for it that their values decrease as n^{-2}.)

$$x - \pi \sim \sum_{n=0}^{\infty} a_n \cos(nx)\,.$$

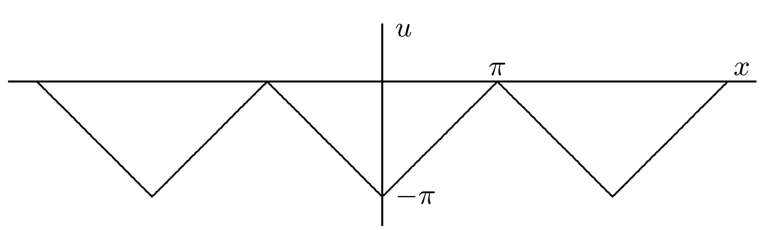
$$a_0 = \frac{1}{\pi}\int_0^{\pi} (x-\pi)\,dx\,,$$

whereas for $n > 0$ we have

$$a_n = \frac{2}{\pi}\int_0^{\pi} (x-\pi)\cos(nx)\,dx\,.$$

(b) Your series represents a periodic function defined on the whole line $-\infty < x < \infty$. Sketch the graph of that function. (Let your x axis run from -3π to 3π.)

The function is even under reflection through each endpoint.



(c) **(honors)**

(1) Does the series converge pointwise?

Yes. (f is piecewise smooth.)

(2) Does it converge uniformly?

Yes. (The periodically extended function is continuous as well as piecewise smooth. Alternatively, $a_n \sim n^{-2}$ is sufficient to guarantee that the series converges uniformly.)

(3) Does it converge in the mean?

Yes. (f is square-integrable.)

(c) **(regular)** Use the series in (a) to solve the heat-conduction problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad (0 < x < \pi,\ -\infty < t < \infty),$$

$$\frac{\partial u}{\partial x}(0,t) = 0\,, \quad \frac{\partial u}{\partial x}(\pi,t) = 0 \qquad (-\infty < t < \infty),$$

$$u(x,0) = f(x) \qquad (0 < x < \pi).$$

(Recall that separation of variables in the heat equation gives the equations

$$X''(x) = -n^2 X(x)\,, \quad T'(t) = -n^2 T(t)\,, \quad X'(0) = 0 = X'(\pi)\,.$$

Don't stop to rederive this.)

$$u(x,t) = \sum_{n=0}^{\infty} a_n \cos(nx) e^{-n^2 t}\,.$$

3. *(37 pts.)* Consider the wave propagation problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad (0 < x < \infty,\ -\infty < t < \infty),$$

$$\frac{\partial u}{\partial x}(0,t) = 0 \qquad (-\infty < t < \infty),$$

with initial data

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x) \qquad (0 < x < \infty).$$

(a) Write down the d'Alembert (also called "method of characteristics") formula for the solution, making sure everything is defined.

Extend f and g so that they are even (under reflection through the origin). (I use the same letters, f and g, for both the original data functions and their extensions.) Let $G(x)$ be the antiderivative of g with $G(0) = 0$. (Then $G(x)$ is an odd function, and $f'(x)$ is also odd.) Now the solution is

$$u(x,t) = \frac{1}{2}[f(x+t) + f(x-t)] + \frac{1}{2}[G(x+t) - G(x-t)]\,.$$

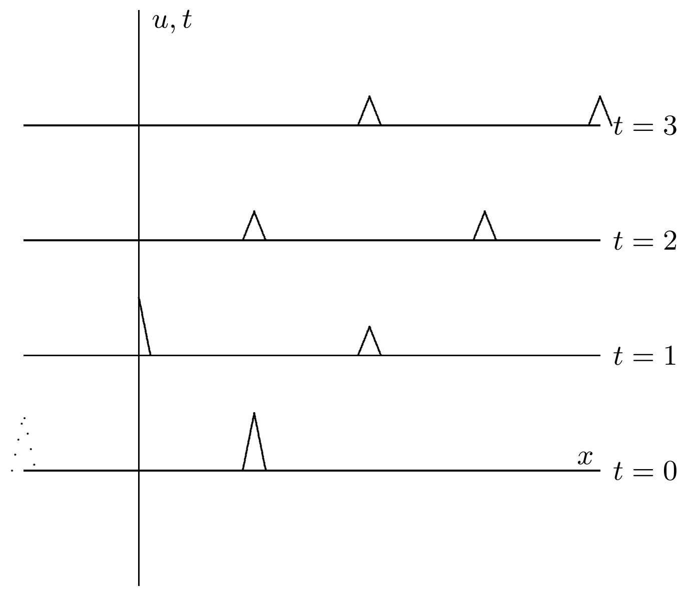
Alternatively, the G terms can be written as

$$\frac{1}{2}\int_{x-t}^{x+t} g(w)\,dw\,.$$

(b) Let $h(x)$ be a nonnegative function sharply peaked around $x = 1$ (for example, $h(x) = e^{-10(x-1)^2}$). Sketch the solution, $u(x,t)$, as a function of x for $t = 0$, $t = 1$, $t = 2$, and $t = 3$, for the initial data

(regular) $u(x,0) = h(x), \quad \dfrac{\partial u}{\partial t}(x,0) = 0$.

The initial blip divides into left-moving and right-moving pulses. With this boundary condition the left-moving pulse does not invert when it reflects; at $t = 1$ the "ghost" pulse from negative x restores it to its original height (but only half its original width).



(honors) $u(x,0) = 0, \quad \dfrac{\partial u}{\partial t}(x,0) = h(x)$.

I'll do this carefully in *Mathematica* and post it in a separate file (along with some intermediate times).

4. *(8 pts.)* Use the complex exponential function to prove that

$$\sin(x+y) = \sin x \cos y + \cos x \sin y\,.$$

The basic definition is

$$e^{\pm ix} = \cos x \pm i \sin x\,.$$

Therefore,

$$\cos x = \frac{1}{2}(e^{ix} + e^{-ix}), \quad \sin x = \frac{1}{2i}(e^{ix} - e^{-ix}).$$

Substitute these into the right side of the identity:

$$\sin x \cos y + \cos x \sin y = \frac{1}{4i}[(e^{ix} - e^{-ix})(e^{iy} + e^{-iy}) + (e^{ix} + e^{-ix})(e^{iy} - e^{-iy})],$$

which simplifies to

$$\frac{1}{2i}(e^{i(x+y)} - e^{-i(x+y)}) = \sin(x+y)\,,$$

as expected.