Math. 460 (Fulling) 9 December 2011

Final Examination – Solutions

1. *(40 pts.)* In 4-dimensional space-time the Riemann tensor, $R_{\alpha\beta\gamma\delta}$, has $4^4 = 256$ components. Show why only 20 of them are independent. (Start by listing the index symmetries.)

The tensor is antisymmetric in its last two indices. $R_{\alpha\beta\delta\gamma} = -R_{\alpha\beta\gamma\delta}$, and symmetric under exchange of the first and last pair of indices, $R_{\gamma\delta\alpha\beta} = R_{\alpha\beta\gamma\delta}$. (It follows that it is antisymmetric in the first two.) Finally, there is a cyclic (Bianchi) symmetry,

$$R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\delta\beta\gamma} = 0.$$

(There are other symmetries that follow from these, but we need not list them.)

Note: It is not correct to say that the consequence of antisymmetry or symmetry of a matrix is just to divide the number of independent matrix elements by 2 . Diagonal and off-diagonal index pairs must be counted separately.

The best way to analyze the problem is to regard R as a 2×2 symmetric matrix whose indices are antisymmetric pairs. The number of independent antisymmetric pairs is $(4 \times 3)/2 = 6$. The number of independent elements of the "big" matrix R therefore is $(6 \times 5)/2 + 6 = 21$ (the number of independent off-diagonal elements plus the number of diagonal elements).

It remains to see how the cyclic symmetry reduces this number. It is easy to see that if any two indices are equal, the cyclic identity is automatically satisfied by virtue of the other symmetries and hence gives no new restriction. (Start by moving one of the equal indices into the α position.) If all four indices are distinct, then the three terms are genuinely distinct with regard to the other symmetries, and we get a new relation that reduces the number of cases from 21 to 20 . Since the other symmetries can be used to put any of the 4 indices in the α position, there is only one such new relation; the number can't be reduced further.

2. *(Essay – 30 pts.)* The Schwarzschild metric is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2.$$

It has mathematical singularities at $r = 2M$ and $r = 0$. Describe (physically and geometrically) what actually happens at those places.

3. *(40 pts.)* [Suppose that] astronomers have discovered that, during some epoch of its history, the expansion of the universe was dominated by a strange substance called "bright stuff" whose equation of state is $p = -\frac{2}{3}\rho$.

   (a) Find $\rho(A)$ and $A(t)$ for this scenario. *Hints:*

$$\frac{d}{dt}(\rho A^3) = -p\,\frac{d}{dt}A^3 .$$

$$\left(\frac{\dot{A}}{A}\right)^2 + \frac{k}{A^2} - \frac{\Lambda}{3} = \frac{8\pi G}{3}\,\rho .$$

What does "dominated" mean?

$$\frac{2\ddot{A}}{A} + \left(\frac{\dot{A}}{A}\right)^2 + \frac{k}{A^2} - \Lambda = 8\pi G p \,.$$

If you can't find $\rho(A)$, assume it is proportional to a power of A and go on to solve for $A(t)$. The conservation law in this case is

$$\frac{d}{dt}(\rho A^3) = \frac{2}{3}\,\rho\,\frac{d}{dt}A^3 \,.$$

If $u = A^3$, then $\frac{d}{dt} = \frac{du}{dt}\frac{d}{du}$ and hence

$$\frac{d}{du}(\rho u) = \frac{2}{3}\,\rho \;\Rightarrow\; u\,\frac{d\rho}{du} = -\frac{1}{3}\rho \;\Rightarrow\; \int \frac{du}{u} = -3\int \frac{d\rho}{\rho} \,.$$

Thus

$$\ln u + c = -3\ln\rho \;\Rightarrow\; \rho = Cu^{-1/3} = CA^{-1} \,.$$

"Dominated" means that in the first Einstein equation we are supposed to neglect the k and Λ terms in comparison with this one. (By the way, the second Einstein equation can be ignored because it adds no information to the first one and the conservation law.) So we have

$$\left(\frac{dA}{dt}\right)^2 = \frac{8\pi GC}{3}\,A^{+1} \;\Rightarrow\; \frac{dA}{dt} \propto A^{1/2} \;\Rightarrow\; \int A^{-1/2}dA \propto \int dt \;\Rightarrow\; 2A^{1/2} \propto t \;\Rightarrow\; A = \text{const.}t^2.$$

(b) Did this epoch come before or after the radiation-dominated epoch? Did it come before or after the epoch of domination by "dark energy"?

This scenario corresponds to $w = -\frac{2}{3}$ in the notation of the lectures. Looking again at the Einstein equation, we see that the cosmological (dark energy) term should dominate at late times, when A is large, whereas the curvature and normal matter terms dominate at earlier times. The radiation term, in particular, goes as A^{-4} and dominates right after the big bang (assuming that the Friedmann–Robertson–Walker picture is valid at all at such early times).

4. *(40 pts.)* In this question, assume that the Christoffel symbols are symmetric in the lower indices.

(a) Write the formula for the covariant derivative of a 1-form (i.e., covariant vector field) in terms of Christoffel symbols.

$$\omega_{\alpha;\beta} = \omega_{\alpha,\beta} - \Gamma^{\gamma}{}_{\alpha\beta}\omega_\gamma \,.$$

(b) One way of defining the Riemann tensor is by the covariant-derivative commutator formula

$$[\nabla_\gamma, \nabla_\delta]\omega_\mu = -R^\nu{}_{\mu\gamma\delta}\,\omega_\nu \,.$$

Use this formula and (a) to derive the formula for $R^\alpha{}_{\beta\gamma\delta}$ in terms of Christoffel symbols.

[Proceed just as in the corresponding calculation for contravariant vector fields, p. 45 of the notes.]

5. *(50 pts.)* The metric (or line element) of the Kerr black hole is

$$ds^2 = -\frac{\Delta - a^2 \sin^2\theta}{\rho^2}\,dt^2 - \frac{4Mar\sin^2\theta}{\rho^2}\,dt\,d\phi + \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\rho^2}\,\sin^2\theta\,d\phi^2 + \frac{\rho^2}{\Delta}\,dr^2 + \rho^2\,d\theta^2\,,$$

where

$$\Delta = r^2 - 2Mr + a^2\,, \qquad \rho^2 = r^2 + a^2\cos^2\theta\,.$$

We can assume that $0 < a < M$ and that $0 < r < \infty$.

(a) Find the *leading behavior* (typically a power of r, times a constant) of each metric coefficient as $r \to \infty$. Comment on what this result says about the physical or geometrical nature of the system described by the Kerr metric.

$$g_{tt} \sim -1\,, \quad g_{t\phi} \sim -\frac{2Ma\sin^2\theta}{r}\,, \quad g_{\phi\phi} \sim r^2\sin^2\theta\,, \quad g_{rr} \sim 1\,, \quad g_{\theta\theta} \sim r^2\,.$$

To leading order (except for the vanishingly small off-diagonal term) this is the metric of flat space-time in spherical coordinates. That is what we expect for an isolated, compact body in flat space. (In fact, if you work to one higher order, to include the $O\left(\frac{1}{r}\right)$ terms, you get the Schwarschild metric.)

(b) Henceforth let's specialize to the plane $\theta = \pi/2$, where the expression reduces to

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 - \frac{4Ma}{r}\,dt\,d\phi + \left(r^2 + a^2 + \frac{2Ma^2}{r}\right)d\phi^2 + \frac{r^2}{r^2 - 2Mr + a^2}\,dr^2.$$

(You are **not** required to rederive this.) Find the value of r where $g_{tt} = 0$. (This surface is called the *ergosphere*.)

$$r = 2M.$$

(c) Find all values of r where $g_{rr} = \infty$, and determine the largest such value that is less than the ergosphere radius from (b). (This surface is an *horizon*, and the region between the horizon and the ergosphere is called the *ergoregion*.) Tabulate the signs of g_{tt}, g_{rr}, and $g_{\phi\phi}$ in the three regions: exterior, ergoregion, and inside the outer horizon.

$$r^2 - 2Mr + a^2 = 0 \;\Rightarrow\; r = M \pm \sqrt{M^2 - a^2}\,.$$

Both roots are in the allowed interval $0 < r < 2M$, and the one with the plus sign is the larger one.

	Interior	Ergoregion	Exterior
g_{tt}	+	+	-
g_{rr}	-	+	+
$g_{\phi\phi}$	+	+	+

(d) Perform a coordinate transformation $\phi = \psi + \Omega t$, where Ω is a constant. (Eliminate ϕ in favor of ψ throughout the line element. The coordinates t and r are unchanged,* but the function g_{tt} multiplying dt^2 will change.) Show that in the new coordinate system g_{tt} becomes positive for large r (in fact, it approaches $+\infty$); explain. *Hint:* Think about a rotating rigid coordinate system in special relativity.

$$d\phi = d\psi + \Omega\, dt \ \Rightarrow\ d\phi^2 = d\psi^2 + 2\Omega\, d\psi\, dt + \Omega^2 dt^2.$$

Therefore,

$$\begin{aligned} ds^2 = {} & \left[\frac{2M}{r} - 1 - \frac{4Ma\Omega}{r} + \Omega^2\left(r^2 + a^2 + \frac{2Ma^2}{r}\right)\right] dt^2 \\ & + \left[-\frac{4Ma}{r} + 2\Omega\left(r^2 + a^2 + \frac{2Ma^2}{r}\right)\right] dt\, d\psi \\ & + \left(r^2 + a^2 + \frac{2Ma^2}{r}\right) d\psi^2 + \frac{r^2}{r^2 - 2Mr + a^2}\, dr^2. \end{aligned}$$

Clearly the leading term in g_{tt} is now $\Omega^2 r^2$, which approaches $+\infty$. In any rigidly rotating reference frame, the rotational motion must become superluminal at sufficiently large radius, simply because the tangential velocity is proportional to Ωr. Thus the basis vector in the time direction becomes spacelike. (This has nothing to do with black holes or gravity; it's just special relativity.)

(e) For a *fixed* value of r, call it r_0, find the value of Ω that renders $g_{t\psi}$ equal to 0. Without doing any more actual calculations, explain why we might be interested in doing this coordinate transformation with the r_0 corresponding to some radial surface inside the ergoregion. Describe the expected sign behavior of (the *new*) g_{tt} for $r \approx r_0$.

$$\Omega = \frac{2Ma}{r_0}\left(r_0^2 + a^2 + \frac{2Ma^2}{r_0}\right)^{-1}.$$

Despite its pathological nature at infinity, such a rotating coordinate system comes close to being an inertial frame at points inside the ergoregion, and therefore it is good for understanding physical processes that take place there. In particular, g_{tt} will now be negative near r_0.

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* For consistency with the notation used in the lecture, one should write (t', r', ϕ') instead of (t, r, ψ). In that notation, the questions in parts (d) and (e) are about $g_{t'\phi'}$ and $g_{t't'}$. The new notation is more convenient for today's purpose.