Math. 460 (Fulling) 12 October 2011

Test A – Solutions

1. *(30 pts.)* Start with a ("contravariant") vector space $\mathcal{V}$.

   (a) Define a $\binom{0}{3}$ tensor in the modern fashion (as a "multilinear functional ... ").

A $\binom{0}{3}$ tensor is a multilinear functional of three vector arguments. That is, if $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathcal{V}$, then $T(\vec{u},\vec{v},\vec{w})$ is a (real) number, and

$$T(\lambda\vec{u}_1 + \vec{u}_2, \vec{v}, \vec{w}) = \lambda T(\vec{u}_1, \vec{v}, \vec{w}) + T(\vec{u}_2, \vec{v}, \vec{w})$$

for all numbers λ and all vectors, with analogous linearity conditions for the other two argument slots.

   (b) Given that under a change of basis the coordinates of a vector in $\mathcal{V}$ transform according to

$$v^{\overline{\alpha}} = \Lambda^{\overline{\alpha}}{}_{\beta}\, v^{\beta} ,$$

   derive the transformation law for the components, $\{T_{\alpha\beta\gamma}\}$, of a $\binom{0}{3}$ tensor. (Start by explaining what "components" are in the context of your definition in (a).)

The tensor components in a particular basis can be defined as the numbers

$$T_{\alpha\beta\gamma} = T(\vec{e}_\alpha, \vec{e}_\beta, \vec{e}_\gamma)$$

obtained by applying the multilinear functional to all possible choices of the vectors from the basis. Equivalently, $\{T_{\alpha\beta\gamma}\}$ are the numbers such that the value of the functional on arbitrary vectors is given by the formula

$$T(\vec{u},\vec{v},\vec{w}) = T_{\alpha\beta\gamma} u^\alpha v^\beta w^\gamma$$

(with the summation convention understood).

Working from the second of these definitions, we can require

$$\begin{aligned} T(\vec{u},\vec{v},\vec{w}) &= T_{\overline{\alpha}\overline{\beta}\overline{\gamma}}\, u^{\overline{\alpha}} v^{\overline{\beta}} w^{\overline{\gamma}} \\ &= T_{\overline{\alpha}\overline{\beta}\overline{\gamma}}\, \Lambda^{\overline{\alpha}}{}_{\alpha} \Lambda^{\overline{\beta}}{}_{\beta} \Lambda^{\overline{\gamma}}{}_{\gamma}\, u^\alpha v^\beta w^\gamma , \end{aligned}$$

from which we read off

$$T_{\alpha\beta\gamma} = T_{\overline{\alpha}\overline{\beta}\overline{\gamma}}\, \Lambda^{\overline{\alpha}}{}_{\alpha} \Lambda^{\overline{\beta}}{}_{\beta} \Lambda^{\overline{\gamma}}{}_{\gamma} .$$

To use the first definition instead, note that the basis change equivalent to the given vector transformation law is

$$\vec{e}_\beta = \Lambda^{\overline{\alpha}}{}_{\beta}\, \vec{e}_{\overline{\alpha}} .$$

Therefore,

$$T(\vec{e}_\alpha, \vec{e}_\beta, \vec{e}_\gamma) = \Lambda^{\overline{\alpha}}{}_{\alpha} \Lambda^{\overline{\beta}}{}_{\beta} \Lambda^{\overline{\gamma}}{}_{\gamma}\, T(\vec{e}_{\overline{\alpha}}, \vec{e}_{\overline{\beta}}, \vec{e}_{\overline{\gamma}}) ,$$

the same result as before. Of course, to get the transformation from unbarred to barred indices, use the inverse matrix for Λ.

2. *(45 pts.)* Let t and x be the usual coordinates in 2-dimensional Minkowski space-time, and define new coordinates u and v by

$$t = u\cosh v\,, \qquad x = u\sinh v\,.$$

(a) If $0 < u < \infty$ and $-\infty < v < \infty$, what region of space-time is actually covered by this coordinate chart?

Since u is always positive, so is t. But x takes either sign. Also, we always have $t > |x|$. In fact, the points reached are precisely those with $t > |x|$, since for such points u and v can always be found.

(b) Find the tangent vectors to the coordinate curves, $\vec{e}_u$ and $\vec{e}_v$.

Take derivatives of x and t with respect to u and v:

$$\vec{e}_u = \begin{pmatrix} \cosh v \\ \sinh v \end{pmatrix}, \qquad \vec{e}_v = \begin{pmatrix} u\sinh v \\ u\cosh v \end{pmatrix}.$$

(c) Find the metric tensor, $\{g_{\alpha\beta}\}$, (or, equivalently, the line element, ds^2) in the new coordinates.

Method 1: Find the matrix of (Lorentz) dot products of the vectors found in (b):

$$\begin{pmatrix} g_{uu} & g_{uv} \\ g_{vu} & g_{vv} \end{pmatrix} = \begin{pmatrix} -\cosh^2 v + \sinh^2 v & -u\sinh v\cosh v + u\sinh v\cosh v \\ -u\sinh v\cosh v + u\sinh v\cosh v & -u^2\sinh^2 v + u^2\cosh^2 v \end{pmatrix}$$
$$= \begin{pmatrix} -1 & 0 \\ 0 & u^2 \end{pmatrix}.$$

Method 2: $ds^2 = -dt^2 + dx^2 =$

$$-(\cosh v\,du + u\sinh v\,dv)^2 + (\sinh v\,du + u\cosh v\,dv)^2 = -du^2 + u^2\,dv^2 + 0\,du\,dv.$$

(d) Find the Christoffel symbols ($\Gamma^u{}_{uv}$, etc.).

Method 1: Use

$$\frac{\partial \vec{e}_\beta}{\partial x^\gamma} = \Gamma^\alpha{}_{\beta\gamma}\,\vec{e}_\alpha\,.$$

$\dfrac{\partial \vec{e}_u}{\partial u} = 0$, so $\Gamma^u{}_{uu} = 0 = \Gamma^v{}_{uu}$.

$\dfrac{\partial \vec{e}_u}{\partial v} = \dfrac{1}{u}\,\vec{e}_v$, so $\Gamma^u{}_{uv} = 0$ and $\Gamma^v{}_{uv} = \dfrac{1}{u}$.

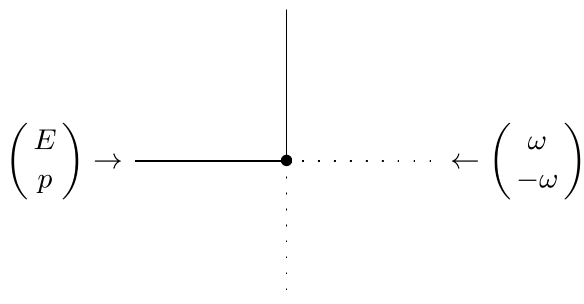
$\dfrac{\partial \vec{e}_v}{\partial u} = \dfrac{1}{u}\,\vec{e}_v$ yields no additional information.

$\dfrac{\partial \vec{e}_v}{\partial v} = u\,\vec{e}_u$, so $\Gamma^u{}_{vv} = u$ and $\Gamma^v{}_{vv} = 0$.

Method 2: Use $\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2}g^{\alpha\mu}(g_{\mu\beta,\gamma} + g_{\mu\gamma,\beta} - g_{\beta\gamma,\mu})$. The results should be the same.

3. *(25 pts.)* Consider the collision of a photon and an electron (mass m). In the center-of-mass frame (and taking $\hbar = 1$, $c = 1$) the photon moves in the negative x direction with energy ω, and the electron moves in the positive x direction with momentum p. Suppose that after the collision the photon is moving exactly in the negative y direction.

   (a) What are the 4-momentum vectors of the outgoing photon and electron in the center-of-mass frame?



Before collision: By definition of center-of-mass frame, $p = \omega$; $E = \sqrt{\omega^2 + m^2}$. Momentum components in the y and z directions are 0 and are not labeled on the diagram.

After collision: By conservation of energy and momentum, the outgoing energies are the same as the incoming ones and the outgoing momenta are the same as the incoming ones except for being in the y direction instead of x direction. (I will prove this claim below and give you 5 points for doing likewise.) Thus the outgoing 4-momenta are

$$\vec{P}_\gamma = \begin{pmatrix} \omega \\ 0 \\ -\omega \\ 0 \end{pmatrix}, \qquad \vec{P}_e = \begin{pmatrix} E \\ 0 \\ \omega \\ 0 \end{pmatrix}.$$

Proof: The electron momentum must be equal and opposite to the photon momentum, so the electron must also be moving along the y axis. Thus the momenta have the form

$$\vec{P}_\gamma = \begin{pmatrix} \omega' \\ 0 \\ -\omega' \\ 0 \end{pmatrix}, \qquad \vec{P}_e = \begin{pmatrix} E' \\ 0 \\ \omega' \\ 0 \end{pmatrix}, \qquad E' = \sqrt{\omega'^2 + m^2}.$$

Conservation of energy says $E' + \omega' = E + \omega$, and since $E + \omega$ is a monotonic function of ω, this equality can hold only when $\omega' = \omega$.

   (b) What is the energy of the outgoing photon in the original rest frame of the (incoming) electron? (Write answer in terms of ω.)

Method 1: The 4-velocity of the electron is the temporal unit vector in its rest frame. In the CM frame this vector is

$$\frac{\vec{P}}{m} = \frac{1}{m} \begin{pmatrix} E \\ \omega \\ 0 \\ 0 \end{pmatrix}.$$

Take its Lorentz dot product with $\vec{P}_\gamma$:

$$-\omega \, \frac{E}{m} = -\,\frac{\omega}{m}\sqrt{\omega^2 + m^2}\,.$$

The outgoing photon energy in the incoming electron rest frame is the negative of this.

Method 2: The Lorentz transformation from the CM frame to the rest frame is

$$\begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where

$$v = \frac{p}{E} = \frac{\omega}{\sqrt{\omega^2 + m^2}} \quad \text{and} \quad \gamma = \frac{1}{\sqrt{1 - v^2}} = \frac{E}{m} = \frac{\sqrt{\omega^2 + m^2}}{m}\,.$$

Apply this to the photon momentum, getting

$$\begin{pmatrix} \gamma\omega \\ -v\gamma\omega \\ -\omega \\ 0 \end{pmatrix}.$$

The energy component is

$$\gamma\omega = \frac{\omega}{m}\sqrt{\omega^2 + m^2}\,.$$