Special Relativity and Electromagnetism

The following problems (composed by Professor P. B. Yasskin) will lead you through the construction of the theory of electromagnetism in special relativity. Please write your response as a connected essay, similar to a chapter of a textbook. If possible, use TEX or a word processor so that you can make revisions easily. See the course handout for due dates. You may consult books and have discussions with other students, but outright copying (except from the problems themselves, when appropriate) is not allowed.

We regard spacetime as the vector space $\mathbf{R}^4$ with a Lorentz-signature metric (pseudo-inner product). Thus, if we choose the orthonormal basis to be

$$e_0 = (1,0,0,0), \quad e_1 = (0,1,0,0), \quad e_2 = (0,0,1,0), \quad e_3 = (0,0,0,1)$$

(so that all indices run from 0 to 3), and the dual basis to be ω^α, then the metric is

$$\eta = \eta_{\alpha\beta}\omega^\alpha \otimes \omega^\beta, \qquad \text{where} \quad \eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and the inverse metric is

$$\eta^{-1} = \eta^{\alpha\beta} e_\alpha \otimes e_\beta, \qquad \text{where} \quad \eta^{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Derivatives will be denoted by $\partial_\alpha = \dfrac{\partial}{\partial x^\alpha}$, and indices will be raised and lowered using η and η^{-1}.

In problems 1–11, we will study the electromagnetic field, which is the 2-form

$$F = F_{\alpha\beta}\,\omega^\alpha \otimes \omega^\beta \qquad \text{where} \quad F_{\alpha\beta} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix},$$

the electromagnetic potential, which is the 1-form

$$A = A_\alpha \omega^\alpha \qquad \text{where} \quad A_\alpha = (\phi, A_1, A_2, A_3),$$

and the electromagnetic current, which is the vector

$$J = J^\alpha e_\alpha \qquad \text{where} \quad J^\alpha = (\rho, J^1, J^2, J^3).$$

1. Show that the rank-3 tensor

$$S_{\alpha\beta\gamma} = \partial_\gamma F_{\alpha\beta} + \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha}$$

is totally antisymmetric, and hence is a 3-form. (There are three pairs of indices to transpose.) Show that this implies that the components of S are all zero except for those for which α, β, and γ are distinct. *Note:* It is not necessary to write out a proof in detail for every choice of the three indices. Start by showing that the formula for S is unchanged when the three indices are subjected to a cyclic permutation.

Similarly, in the next few parts much writing can be saved by noting that the input formulas are symmetric under cyclic permutations of the spatial indices ($x \to y \to z \to x$).

2. Write out the components of the equations

$$\partial_\gamma F_{\alpha\beta} + \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} = 0 \qquad \text{and} \qquad \partial_\beta F^{\alpha\beta} = 4\pi J^\alpha$$

to see that these are the Maxwell equations

$$\begin{aligned} &\vec{\nabla} \cdot \vec{B} = 0, && \vec{\nabla} \cdot \vec{E} = 4\pi\rho, \\ &\vec{\nabla} \times \vec{E} + \partial_t \vec{B} = 0, && \vec{\nabla} \times \vec{B} - \partial_t \vec{E} = 4\pi \vec{J}. \end{aligned}$$

3. Write out the components of the equations

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$$

to find expressions for $\vec{E}$ and $\vec{B}$ in terms of ϕ and $\vec{A}$. (*Note:* ϕ is the negative of the usual scalar potential.)

4. Show (in 4-dimensional notation) that if

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$$

is satisfied, then

$$\partial_\gamma F_{\alpha\beta} + \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} = 0$$

is automatically satisfied. (*Note:* The 3-dimensional version of these equations is a pair of identities: $\vec{\nabla} \cdot \vec{\nabla} \times \vec{A} = 0$ and $\vec{\nabla} \times \vec{\nabla}\phi = 0$.) Consequently, this subset of the Maxwell equations is actually an identity; it is sometimes referred to as the electromagnetic Bianchi identity.

5. Substitute

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$$

into the remaining Maxwell equation

$$\partial_\beta F^{\alpha\beta} = 4\pi J^\alpha$$

to obtain the Maxwell equation for A_α. *Note:* Part 7 will be much easier if you carry out Part 5 entirely in 4-dimensional notation.

6. Let χ be a function. Also let

$$A'_\alpha = A_\alpha + \partial_\alpha \chi \qquad \text{and} \qquad F'_{\alpha\beta} = \partial_\alpha A'_\beta - \partial_\beta A'_\alpha \,.$$

   This is called a gauge transformation. Relate $F'_{\alpha\beta}$ to $F_{\alpha\beta}$ to see that the electromagnetic field is gauge-invariant.

7. The equation of motion you found above for A_α can be simplified by a gauge transformation. We will use (twice) the fact that a wave equation ($\partial^\alpha \partial_\alpha \psi = \sigma$) with an arbitrary but specified source, σ, always has a solution, ψ (not unique). (Perhaps you can find a reference for this theorem?) Given A_α, show that there always exists a function χ such that A'_α satisfies

$$\partial^\alpha A'_\alpha = 0.$$

   This gauge is called *Lorenz gauge*. Observe then that the components of the Maxwell equation for A'_α are wave equations, and conclude that they always have solutions for any arbitrary but specified current J^α.

8. Write out the components of the equation

$$\partial_\alpha J^\alpha = 0$$

   to see that this is conservation of electric charge. Show that if

$$\partial_\beta F^{\alpha\beta} = 4\pi J^\alpha$$

   is satisfied, then

$$\partial_\alpha J^\alpha = 0$$

   is automatically satisfied. This is sometimes referred to as an automatic conservation law. (*Hint:* If $A_{\mu\nu}$ is an expression symmetric in its indices and $B^{\mu\nu}$ is antisymmetric, then one line of index algebra shows that $A_{\mu\nu}B^{\mu\nu} = 0$.)

9. Write out the function

$$\mathcal{L} = -\tfrac{1}{4} F^{\gamma\delta} F_{\gamma\delta}$$

   in terms of $\vec{E}$ and $\vec{B}$. This function is called the Lagrangian density for the vacuum electromagnetic field. It is sometimes interpreted as the difference between the kinetic energy $\frac{1}{2}|\vec{E}|^2$ and the potential energy $\frac{1}{2}|\vec{B}|^2$. Also write out the Lagrangian density in terms of ϕ and $\vec{A}$.

10. Write out the components of the tensor

$$T^{\alpha\beta} = \frac{1}{4\pi}\left(F^{\alpha\gamma}F^{\beta}{}_{\gamma} - \tfrac{1}{4}\eta^{\alpha\beta}F^{\gamma\delta}F_{\gamma\delta}\right)$$

in terms of $\vec{E}$ and $\vec{B}$, to see that this consists of the electromagnetic energy density, momentum density, energy current, and momentum current (or stress). This tensor is called the Maxwell energy-momentum-stress tensor.

11. In 4-dimensional notation, compute the divergence of the energy-momentum tensor and use the Bianchi identities and the Maxwell equations to show that

$$\partial_\beta T^{\alpha\beta} = -J_\beta F^{\alpha\beta}.$$

Problem 11 shows that the electromagnetic energy-momentum is not conserved if the current is nonzero. The reason for this is that we have ignored the energy-momentum of the charged particles producing the current. In problem 12, we study the motion of a charged particle. Then in problem 13 we study the energy-momentum tensor of a fluid of charged particles.

12. A particle of mass m with electric charge q is moving on the parametrized path $x^\alpha(\tau)$ where τ is the proper time. Consequently, it has unit timelike tangent vector

$$U = U^\alpha e_\alpha, \qquad \text{where} \quad U^\alpha = \frac{\partial x^\alpha}{\partial \tau} = (\gamma, \gamma v^1, \gamma v^2, \gamma v^3)$$

and where

$$\gamma = \frac{1}{\sqrt{1-|\vec{v}|^2}}\,.$$

Further, its 4-momentum is

$$p^\alpha = mU^\alpha.$$

Write out the components of the equations

$$U^\beta \partial_\beta(p^\alpha) = q\,U^\beta F^\alpha{}_\beta$$

to obtain the Lorentz force and power laws. (*Hints:* Don't expand p^α. Be careful with the factors of γ.)

13. Consider a fluid of charged particles of rest mass m and charge q, with fluid velocity U^α and energy density ρ in the instantaneous local rest frame. Then the charge density in the instantaneous local rest frame is $(q/m)\rho$ and the electromagnetic current is

$$J^\alpha = \frac{q}{m}\,\rho\, U^\alpha .$$

Assuming that the particles are non-interacting except for their electromagnetic forces, then

(i) the energy momentum tensor for the fluid is that for dust:

$$T^{\alpha\beta}_{\text{fluid}} = \rho\, U^\alpha U^\beta ,$$

(ii) each particle moves according to the Lorentz force equation:

$$U^\beta \partial_\beta (m U^\alpha) = q\, U^\beta F^\alpha{}_\beta \, ,$$

(iii) the energy momentum tensor for the electromagnetic field is

$$T^{\alpha\beta}_{\text{em}} = \frac{1}{4\pi}\left(F^{\alpha\gamma}F^\beta{}_\gamma - \tfrac{1}{4}\eta^{\alpha\beta}F^{\gamma\delta}F_{\gamma\delta}\right),$$

(iv) and the electromagnetic field satisfies the Bianchi identities and the Maxwell equations with current J^α.

Then, as seen in problem 8, the electromagnetic current is conserved:

$$\partial_\alpha J^\alpha = 0,$$

and, as seen in problem 11, the electromagnetic energy-momentum tensor satisfies

$$\partial_\beta T^{\alpha\beta}_{\text{em}} = -J_\beta F^{\alpha\beta}.$$

Now use the Lorentz force equation and the conservation of electromagnetic current to show that the fluid energy-momentum tensor satisfies

$$\partial_\beta T^{\alpha\beta}_{\text{fluid}} = J_\beta F^{\alpha\beta}.$$

(*Hint:* Factor $T^{\alpha\beta}_{\text{fluid}}$ as

$$T^{\alpha\beta}_{\text{fluid}} = \left(U^\alpha\right)\left(\rho U^\beta\right)$$

and use the product rule.) Thus the total energy-momentum is conserved:

$$\partial_\beta\left(T^{\alpha\beta}_{\text{fluid}} + T^{\alpha\beta}_{\text{em}}\right) = 0.$$

In problems 14 and 15, we study the behavior of the electromagnetic field under rotations and Lorentz boosts. Under a general Lorentz transformation, $\Lambda^{\alpha'}{}_{\gamma}$, the electromagnetic field transforms according to

$$F_{\alpha'\beta'} = F_{\gamma\delta}(\Lambda^{-1})^{\gamma}{}_{\alpha'}(\Lambda^{-1})^{\delta}{}_{\beta'}\,.$$

We then write

$$F_{\gamma\delta} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & B^3 & -B^2 \\ E^2 & -B^3 & 0 & B^1 \\ E^3 & B^2 & -B^1 & 0 \end{pmatrix} \quad \text{and} \quad F_{\alpha'\beta'} = \begin{pmatrix} 0 & -E^{1'} & -E^{2'} & -E^{3'} \\ E^{1'} & 0 & B^{3'} & -B^{2'} \\ E^{2'} & -B^{3'} & 0 & B^{1'} \\ E^{3'} & B^{2'} & -B^{1'} & 0 \end{pmatrix}.$$

14. First assume that the Lorentz transformation is a rotation about the z-axis:

$$\Lambda^{\alpha'}{}_{\gamma} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R & \\ 0 & & & \end{pmatrix} \qquad \text{where} \quad R^{i'}{}_{j} = \begin{pmatrix} \cos\omega & \sin\omega & 0 \\ -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Show that $\vec{E}$ and $\vec{B}$ transform as vectors:

$$E^{i'} = R^{i'}{}_{j}E^{j} \qquad \text{and} \qquad B^{i'} = R^{i'}{}_{j}B^{j}.$$

Show that this generalizes to arbitrary rotation matrices. (Try to give a conceptual argument, not a grubby calculation.)

15. Now assume that the Lorentz transformation is a boost in the z-direction with velocity $\vec{v} = v\hat{e}_z$:

$$\Lambda^{\alpha'}{}_{\gamma} = \begin{pmatrix} \cosh\lambda & 0 & 0 & \sinh\lambda \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\lambda & 0 & 0 & \cosh\lambda \end{pmatrix}$$

where $\cosh\lambda = \gamma = \dfrac{1}{\sqrt{1-v^2}}$ and $\sinh\lambda = \gamma v = \dfrac{v}{\sqrt{1-v^2}}$. Find expressions for $\vec{E}'$ and $\vec{B}'$ in terms of $\vec{E}$ and $\vec{B}$ and either λ or v.

In problems 16 and 17, we study the Lagrangian and Hamiltonian formulations of electromagnetism. Each problem begins with a discussion of the analogous formulation of classical mechanics and the situation for a general field theory with fields ψ^A, for $A = 1, \ldots, N$. Then the special case of electromagnetism is treated with ψ^A replaced by A_α.

16. In classical particle mechanics, the Lagrangian is

$$L = T - V = \tfrac{1}{2}m|\vec{v}|^2 - V(\vec{x}).$$

In discussing this Lagrangian, it is useful to regard $\vec{x}$ and $\vec{v} = \dfrac{d\vec{x}}{dt}$ as independent variables. One then computes

$$\frac{\partial L}{\partial x^i} = -\partial_i V \qquad \text{and} \qquad p_i = \frac{\partial L}{\partial v^i} = mv_i\,.$$

(The quantity p_i is called the momentum conjugate to x^i.) Then the Euler–Lagrange equations for this Lagrangian are

$$\frac{d}{dt}\frac{\partial L}{\partial v^i} - \frac{\partial L}{\partial x^i} = 0,$$

or

$$\frac{d}{dt}(mv_i) + \partial_i V = 0,$$

which is Newton's equation with the force identified as $F_i = -\partial_i V$, the gradient of the potential.

Similarly, in field theory, in discussing a Lagrangian density, $\mathcal{L}$, it is useful to regard the fields ψ^A and their derivatives $\partial_\alpha \psi^A$ as independent variables. One then computes

$$\frac{\partial \mathcal{L}}{\partial \psi^A} \quad \text{and} \quad \pi_A{}^\alpha = \frac{\partial \mathcal{L}}{\partial \partial_\alpha \psi^A}\,.$$

(The quantity $\pi_A \equiv \pi_A{}^0$ is the conjugate momentum to ψ^A, while the 4-vector $\pi_A{}^\alpha$ is sometimes called the conjugate multimomentum to ψ^A .) Then the Euler–Lagrange equations for the Lagrangian density, $\mathcal{L}$, are

$$\partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial \partial_\alpha \psi^A} \right) - \frac{\partial \mathcal{L}}{\partial \psi^A} = 0.$$

Both sets of Euler–Lagrange equations given above can be derived from appropriate variational principles. In this exercise, we apply the field theory version to the vacuum Maxwell Lagrangian density,

$$\mathcal{L} = -\tfrac{1}{4}F^{\gamma\delta}F_{\gamma\delta}\,.$$

Compute

$$\frac{\partial \mathcal{L}}{\partial A_\alpha} \qquad \text{and} \qquad \pi^{\alpha\beta} = \frac{\partial \mathcal{L}}{\partial \partial_\beta A_\alpha}\,.$$

Identify the conjugate momenta to $A_0 = \phi$ and to A_i :

$$\pi^\alpha \equiv \pi^{\alpha 0} = \frac{\partial \mathcal{L}}{\partial \partial_0 A_\alpha} \,.$$

Compute the Euler–Lagrange equations

$$\partial_\beta \left(\frac{\partial \mathcal{L}}{\partial \partial_\beta A_\alpha} \right) - \frac{\partial \mathcal{L}}{\partial A_\alpha} = 0$$

and verify that these are the vacuum Maxwell equations.

Hints: Explicitly write out all metrics in $\mathcal{L}$ (but do not use α or β as dummy indices). At the first step, do not expand F in terms of derivatives of A. Use the chain rule. Then compute the derivatives of F using formulas such as

$$\frac{\partial \partial_\delta A_\gamma}{\partial \partial_\beta A_\alpha} = \delta_\beta^\delta \delta_\alpha^\gamma \,.$$

17. In classical mechanics, the Hamiltonian is

$$H = p_i v^i - L(\vec{x}, \vec{v})$$

but expressed as a function of $\vec{x}$ and $\vec{p}$:

$$H = \frac{\vec{p}^{\,2}}{m} - \left[\frac{\vec{p}^{\,2}}{2m} - V(\vec{x}) \right] = \frac{\vec{p}^{\,2}}{2m} + V(\vec{x}) = T + V.$$

Similarly, for a field theory, the Hamiltonian density is

$$\mathcal{H} = \pi_A \partial_0 \psi^A - \mathcal{L}(\psi^A, \partial_\alpha \psi^A),$$

but expressed as a function of ψ^A, $\partial_i \psi^A$, and π_A. For electromagnetism with no sources ($J^\alpha = 0$), express the Hamiltonian density

$$\mathcal{H} = \pi^\alpha \partial_0 A_\alpha - \mathcal{L}(A_\alpha, \partial_\beta A_\alpha)$$

as a function of A_α, $\partial_i A_\alpha$, and the non-zero components of π^α. Then express $\mathcal{H}$ as a function of $\vec{E}$ and $\vec{B}$ to see whether the Hamiltonian density, $\mathcal{H}$, is equal to the energy density, T^{00}. If not (SPOILER ALERT!), show that nevertheless their integrals give the same total energy, if the fields fall off fast enough at infinity.