Math. 460 (Fulling) 9 December 2019

Final Examination – Solutions

1. *(40 pts.)* In 4-dimensional space-time the Riemann tensor, $R_{\alpha\beta\gamma\delta}$, has $4^4 = 256$ components. Show why only 20 of them are independent.

   (Start by listing the index symmetries of the Riemann tensor. Treat R as a square matrix whose indices are antisymmetric pairs of space-time indices.)

There are three basic symmetries:

$$R_{\alpha\beta\delta\gamma} = -R_{\alpha\beta\gamma\delta} \quad \text{(antisymmetry)},$$
$$R_{\gamma\delta\alpha\beta} = R_{\alpha\beta\gamma\delta} \quad \text{(pair symmetry)},$$
$$R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\delta\beta\gamma} = 0 \quad \text{(cyclic symmetry)}.$$

Others follow from these; in particular, R is antisymmetric in the first index pair as well.

Given the antisymmetry (and dimension $= 4$), there are $\frac{4\times 3}{2} = 6$ independent index pairs, each containing distinct indices with order not mattering. With respect to these superindices, R is a symmetric matrix, with

$$6(\text{diagonal}) + \frac{6\times 5}{2} = 21$$

independent elements. It remains to investigate the effects of the cyclic symmetry. If two indices are equal, say α and β, the identity has only two terms,

$$R_{\alpha\gamma\delta\alpha} + R_{\alpha\delta\alpha\gamma} = 0\,.$$

But this follows from the other two symmetries and thus tells us nothing new. If the indices are distinct (which can happen in only one essentially different way in dimension 4), the identity allows one permutation to be replaced by the sum of two others. Therefore, there are exactly 20 independent components (in a generic space-time).

2. *(10 pts.)* Consider a metric of the form

$$ds^2 = -dt^2 + C(x)^2\,dx^2 + dy^2 + dz^2.$$

   Here $C(x)$ is a function only of x. Show that this space-time is (at least locally) flat!

   (Don't calculate a Riemann tensor. If you need to spend more than 2 minutes on this question, you have missed the point.)

Introduce a new coordinate by $dw = C(x)\,dx$ (implemented by $w = \int C(x)\,dx$). Then

$$ds^2 = -dt^2 + dw^2 + dy^2 + dz^2,$$

which is the metric of ordinary special-relativistic Minkowski space.

3. *(40 pts.)* Recall that the main dynamical equation for a homogeneous, isotropic universe is

$$\left(\frac{\dot a}{a}\right)^2 + \frac{k}{a^2} - \frac{\Lambda}{3} = \frac{8\pi G}{3}\,\rho\,.$$

   (Here $a(t)$ is the Robertson–Walker scale factor, loosely called "radius of the universe", which is sometimes denoted $R(t)$.) G, k, and Λ are constants (G positive).

   (a) Explain what k is.

k represents the *spatial* curvature of the universe (as a three-dimensional manifold at each fixed time). It is positive if the space is a three-dimensional sphere, zero if the space is flat, and negative if the space is hyperboloidal (like a saddle). When k is not zero, a can be rescaled to make $|k| = 1$.

(b) Suppose that $\rho = Ca^{-4}$. (C is a positive constant.) What sort of matter predominates in this universe, and what is the corresponding pressure?

radiation (massless particles), for which $p = \frac{1}{3}\rho$.

(c) Suppose that ρ is as in (b) and that $k > 0$. Find a Λ capable of making a independent of time.

If $\dot{a} = 0$, the Einstein equation becomes

$$\frac{k}{a^2} - \frac{\Lambda}{3} = \frac{8\pi GC}{3} a^{-4}.$$

Solve:

$$\Lambda = \frac{3k}{a^2} - \frac{8\pi GC}{a^4}.$$

So, for any a_0 there is a Λ that makes $a(t) = a_0$ a solution.

(d) Show that the other Einstein equation,

$$\frac{2\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} - \Lambda = -\,8\pi Gp\,,$$

unexpectedly provides an additional constraint among k, C, Λ, and a in this static situation.

In our case this equation becomes just

$$\Lambda = \frac{k}{a^2} + \frac{8\pi GC}{3a^4}.$$

Combined with the result of (c), this implies

$$\frac{2k}{a^2} = \frac{32\pi GC}{3a^4},$$

or

$$a = \sqrt{\frac{16\pi GC}{3k}}.$$

Thus there is one less degree of freedom in the parameters of the model than we thought in (c). (The standard proof that the second Einstein equation adds no information to the first involves cancelling a factor $\dot{a}/a$, so it is not valid for a static solution.)

4. *(Essay – 30 pts.)* The Schwarzschild metric is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2.$$

It has mathematical singularities at $r = 2M$ and $r = 0$. Describe (physically and geometrically) what actually happens at those places.

5. *(40 pts.)* Do **ONE** of these [(A) or (B)]. **If you try both, clearly indicate which one you want graded.**

(A) Recall that under a non-Abelian gauge transformation, $U(x)$, a connection form transforms by the law

$$\tilde{w}_\mu = U w_\mu U^{-1} - (\partial_\mu U) U^{-1} \ \textbf{or}\ U(w_\mu - U^{-1}\partial_\mu U)U^{-1},$$

and also that the gauge field strength is defined by

$$Y_{\mu\nu} = w_{\nu,\mu} - w_{\mu,\nu} + [w_\mu, w_\nu].$$

Show that under a gauge transformation, Y transforms as a gauge tensor:

$$\tilde{Y}_{\mu\nu} = U(x) Y_{\mu\nu}(x) U(x)^{-1}.$$

Hint: $\partial_\mu(U^{-1}) = -U^{-1}(\partial_\mu U)U^{-1}$.

[See either qu. 4 of 2008 final or qu. 4(A) of 2013 final.]

(B)

(a) Show that if a vector field ξ^α satisfies Killing's equation,

$$\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0,$$

then

$$p_\alpha \xi^\alpha = \text{constant}$$

along the geodesic with tangent vector $\vec{V} = \dfrac{\vec{p}}{m}$.

[See qu. 2(a) of 2009 Test B. The rest of that question is equivalent to the rest of this one, but I prefer to write out new solutions for the latter.]

(b) Show that if the metric coefficients $g_{\mu\nu}$ are all independent of a particular coordinate x^β, then the vector field $\xi^\alpha = \delta^{\alpha\beta}$ satisfies Killing's equation.

Because the covariant derivative of the metric is 0,

$$\nabla_\alpha \xi_\beta = g_{\beta\gamma} \nabla_\alpha \xi^\gamma = g_{\beta\gamma} \partial_\alpha \xi^\gamma + g_{\beta\gamma} \Gamma^\gamma_{\delta\alpha} \xi^\delta.$$

Here the first term is 0 because ξ^γ is constant, and the second is, by the formula for Γ,

$$\tfrac{1}{2}\big(g_{\beta\alpha,\delta} + g_{\beta\delta,\alpha} - g_{\delta\alpha,\beta}\big)\xi^\delta.$$

Here the last two terms together are antisymmetric in α and β, while the first term is symmetric. Therefore,

$$\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = g_{\alpha\beta,\delta} \xi^\delta = g_{\alpha\beta,\beta} = 0$$

because g is independent of β.

(c) What conclusion can you draw by combining (a) and (b)?

If the metric is independent of x^β , then the conjugate momentum p_β is conserved.

6. *(Essay – 40 pts.)* Tell me what you know about **TWO** of these topics.

(A) The independent degrees of freedom of the electromagnetic and gravitational fields.

(B) Parallel transport and its relation to curvature. (Emphasize geometrical concepts rather than trying to reconstruct equations.)

(C) Lagrangians and variational principles for geodesics.

(D) The *equation of geodesic deviation*,

$$\frac{D^2 w^\alpha}{d\lambda^2} = R^\alpha{}_{\mu\nu\beta}\, u^\mu u^\nu w^\beta \,.$$

(Explain what the symbols mean and what the equation has to do with tides.)

(E) The Penrose–Floyd process of energy extraction from a rotating black hole.