# Tensors and General Relativity

## Mathematics 460



version of Fall 2019

# Introductory Remarks

## Is this a math course or a physics course?

General relativity is taught in the mathematics department at the undergraduate level (as well as the physics department at the graduate level) because –

- There are faculty members in the math department doing research in areas related to general relativity.

- An introductory GR course requires a large dose of special mathematics, not encountered in other branches of physics at the undergraduate level (tensors, manifolds, curvature, covariant derivatives). Many of these *do* have modern applications outside relativity, however.

- You asked for it. Undergraduate physics majors agitated for the course on a fairly regular basis until both departments were convinced that it should be offered on a biennial basis (alternating with Math 439, Differential Geometry of Curves and Surfaces, in the fall).

This course will do very little with observational astrophysics, gravitational wave detectors, etc. That is left for the physics department (whose full name is now Physics and Astronomy!).

Schutz's (formerly) green book (now blue) is a physicist's book, but with a good treatment of the math; in particular, an excellent pedagogical balance between modern abstract language and classical "index" notation.

We will follow Schutz's book closely, skipping Chapter 9 (gravitational waves) and downplaying Chapter 4 (fluids). There will be external material on electromagnetism (you do the work) and on gauge field theories as another application of covariant derivatives (I do the work).

$$\nabla_\mu = \frac{\partial}{\partial x^\mu} + \Gamma^\rho_{\nu\mu} \quad \text{vs.} \quad D_\mu = \frac{\partial}{\partial x^\mu} - ieA_\mu \,.$$

We need to cover about a chapter a week on average, but some will take more time than others.

The main business of the first $1\frac{1}{2}$ weeks will be quick reviews of special relativity and vectors, Chapters 1 and 2. For these chapters systematic lectures are not possible; we'll work mostly with exercises. Try to read Chapter 1 by next time (hard part: Secs. 1.5 and 1.6). Be prepared to ask questions or work problems.

Usually, I will lecture only when I have something to say in addition to what's in the book. Especially at the beginning, considerable time will be devoted to class discussion of exercises. There is, or will be, a list of exercises on the class web page. Not all of the exercises will be collected and graded. As the semester proceeds and the material becomes less familiar, the balance in class will shift from exercises to lecturing, and there will be more written work (longer exercises, but many fewer of them).

## Some other books

*short* *physics* $\longleftrightarrow$ *math*

Berry
QB981.B54

Rindler
QC173.55.R56.1977

gray Schutz
QC207.D52.S34

easy O'Neill
QA641.05

D'Inverno
QC173.55.D56.1992

Lovett
QA649.L68.2010

Hartle

Walecka
QC173.6.W36.2007

Bishop–Goldberg
A433.B54

Narlikar
QC173.6.N369.2010

$\updownarrow$

Ohanian
QC178.O35

Stephani
QC178/S8213.1990

Frankel
QC173.55.F7

Dodson–Poston
QA649.D6.1990

Adler–Bazin–Schiffer
QC173.6.A34.1975

Burke
QC20.7.D52.B87.1985

Carroll
QC173.6.H63.2006eb

big Frankel
QC20.F7.2004

Isham
QA641.I84.1999

big Will
QC178.W47

Weinberg
QC6.W47

Grøn–Hervik
QC173.55.O55.2007eb

Misner–Thorne–Wheeler
QC178.M57

Wald
QC173.6.W35.1984

big O'Neill
QA3.P8.v103

Hobson-Ef.-L.
QC173,6.H63.2006eb

Zel'dovich–Novikov
QB461.Z4413

Poisson
QC173.6.P65.2004

Choquet–DeWitt–D.
QC20.7.A5.C48.1982

# Review of Special Relativity *(Chapter 1)*

**Recommended supplementary reading:** J. R. Newman, "Einstein's Great Idea", in *Adventures of the Mind*, ed. by R. Thruelsen and J. Kobler (Knopf, New York, 1959), pp. 219–236. CB425.S357. Although "popular", this article presents a number of the important ideas of relativity very well.

(While on the subject of popularizations, I want to mention that one of the best recent popular books on relativity is C. Will, *Was Einstein Right?* (Basic Books, New York, 1986); QC173.6.W55.1986. The emphasis is on experimental and observational tests, especially those that are still being carried out. Will also has a more technical book on that topic.)

G. Holton, "Einstein and the 'crucial' experiment", *Amer. J. Phys.* **37** (1969) 968. Cf. Schutz p. 2: "It is not clear whether Einstein himself was influenced by [the Michelson–Morley experiment]." Einstein wrote, "In my personal struggle

Michelson's experiment played no role or at least no decisive role." But when he was writing in general terms about the justification of special relativity or its place in physics, he would mention the experiment.

We must emphasize the geometrical viewpoint (space-time).

Space-time diagrams are traditionally drawn with time axis vertical, even though a particle path is $x = f(t)$. Thus the slope of the worldline of a particle with constant velocity $v$ is $1/v$.

**Natural units:** We take the speed of light to be $c = 1$. For material bodies, $v < 1$.

$$[\text{time}] = [\text{length}].$$

Later we may also choose

$$\hbar = 1 \qquad [\text{mass}] = [\text{length}]^{-1}$$

or

$$G = 1 \qquad [\text{mass}] = [\text{length}]$$

or both (all quantities dimensionless).

## Inertial observer = Cartesian coordinate system = Frame

An idealized observer is "someone who goes around collecting floppy disks" (or flash drives?) from a grid of assistants or instruments. Cf. M. C. Escher's etching, "Depth". This conception avoids the complications produced by the finite speed of light if one tries to identify the empirical "present" with what a human observer "sees" at an instant of time. (The latter corresponds to conditions on a light cone, not a time slice.)

Here we are walking into a notorious philosophical issue: how empirical is (or should be) physics? Einstein is quoted as saying that in theoretical physics we make ourselves *pictures* of the world, and these are free creations of the human mind. That is, sound science must be consistent with experiment, it must be testable by experiment, but it is not merely a summary of sensory data. We believe in physical objects, not just perspective views (permanent, rectangular windows, not fleeting trapezoids).

An operational definition of the time slices for an inertial observer is analogous to the construction of a perpendicular bisector in Euclidean geometry: We demand equal times for the transmission and reflection of light pulses from the "events" in question. (See Schutz, Sec. 1.5.)

However, this association of frames with real observers must not be taken too literally. Quotation from E. Schrödinger, *Expanding Universes* (Cambridge U. P., 1957), p. 20:

> [T]here is no earthly reason for compelling anybody to change the frame of reference he uses in his computations whenever he takes a walk. ... Let me

on this occasion denounce the abuse which has crept in from popular *exposés*, viz. to connect any particular frame of reference ... with the behaviour (motion) of him who uses it. The physicist's whereabouts are his private affair. It is the very gist of relativity than anybody may use any frame. Indeed, we study, for example, particle collisions alternately in the laboratory frame and in the centre-of-mass frame without having to board a supersonic aeroplane in the latter case.

## References on the twin (clock) paradox


1. E. S. Lowry, The clock paradox, *Amer. J. Phys.* **31**, 59 (1963).

2. C. B. Brans and D. R. Stewart, Unaccelerated-returning-twin paradox in flat space-time, *Phys. Rev. D* **8**, 1662–1666 (1973).

3. B. R. Holstein and A. R. Swift, The relativity twins in free fall, *Amer. J.*

*Phys.* **40**, 746–750 (1972).

Each observer expects the other's clock to run slow by a factor

$$\frac{1}{\gamma} = \sqrt{1-\beta^2} \qquad \left(\beta \equiv \frac{v}{c} = v\right).$$

One should understand what is wrong with each of these canards:

1. "Relativity says that all observers are equivalent; therefore, the elapsed times must indeed be the same at the end. If not, Einstein's theory is inconsistent!"

2. "It's an acceleration effect. Somehow, the fact that the 'younger' twin accelerates for the home journey makes all the difference."

And here is another topic for class discussion:

3. Explain the apparent asymmetry between time *dilation* and length *contraction*.

### Miscellaneous remarks

1. In relativity and differential geometry we normally label coordinates and the components of vectors by superscripts, not subscripts. Subscripts will mean something else later. These superscripts must not be confused with exponents! The difference is usually clear from context.

2. In the metric (1.1), whether to put the minus sign with the time or the space terms is a matter of convention. Schutz puts it with time.

3. Space-time geometry and Lorentz transformations offer many analogies with Euclidean geometry and rotations. See Problem 19 and Figures 1.5 and 1.11,

etc. One thing which has no analogue is the *light cone*.

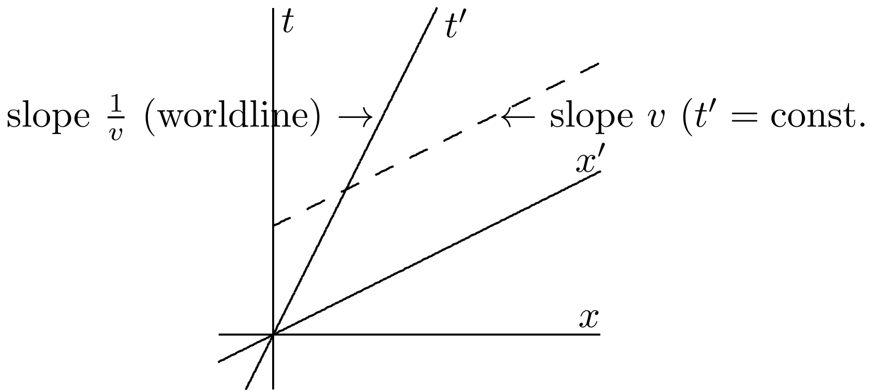
4. When the spatial dimension is greater than 1, the most general Lorentz transformation involves a rotation as well as a "boost". The composition of boosts in two different directions is not a pure boost. (Details later.)

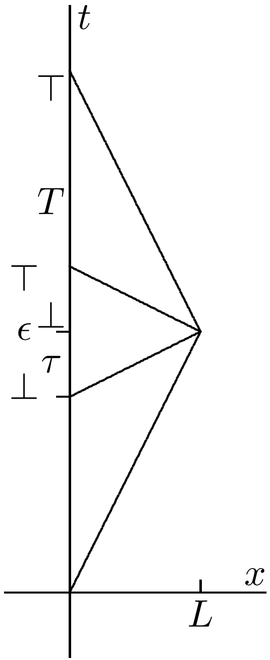
5. There are various assumptions hidden in Sec. 1.6. Most seriously, the author assumes that the natural coordinates of two inertial frames are connected by a *linear* relation. The effect is to exclude curved space-times of maximal symmetry, called *de Sitter spaces*. If you have learned something about Lie algebras, I recommend

    H. Bacry and J.-M. Lévy-Leblond, Possible kinematics, *J. Math. Phys.* **9**, 1605–1614 (1968).

## SOME DETAILS ABOUT THE TWIN PARADOX

First review the standard situation of two frames in relative motion at speed $v$. The $t'$ axis (path of the moving observer) has slope $1/v$. The $x'$ axis and all other equal-time hypersurfaces of the moving observer have slope $v$.






Second, consider the standard twin scenario as graphed by Lowry (and Schutz in the appendix to Chapter 1).

Let the starting point be $t = t' = 0$, $x = x' = 0$, and let the point of return be $(0, t)$ in the stationary frame. The stationary observer attributes a time dilation to the moving clock:

$$t' = \frac{t}{\gamma}$$

where $\gamma = (1 - v^2)^{-1/2} > 1$. The moving observer attributes to the stationary clock a similar dilation plus a gap:

$$t = \frac{t'}{\gamma} + \epsilon.$$

Let's calculate $\epsilon$: Consistency requires

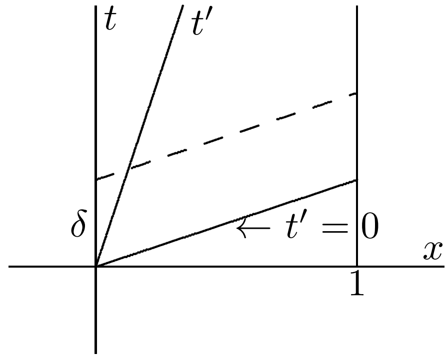
$$\epsilon = t - \frac{t'}{\gamma} = t\left(1 - \frac{1}{\gamma^2}\right) = v^2 t,$$

since

$$\frac{1}{\gamma^2} = 1 - v^2.$$

To see this a different way, let $T = t/2$ and observe that the distance traveled outward is $L = vT$. Therefore, since $t' = \text{const}$ surfaces have slope $v$. the half-gap is $\tau = vL = v^2 T$. Thus $\epsilon = 2\tau = 2v^2 T = v^2 t$, as claimed.

Third, consider the Brans–Stewart model with circumference 1.




Here the dashed line is a natural continuation of the line $t' = 0$. That line and all its continuations are the closest thing we have to an $x'$ axis in this situation. Label the spacing on the $t$ axis between the solid and dashed lines as $\delta$.

Follow the moving observer (whose worldline is the $t'$ axis) around the cylinder back to the starting point (the $t$ axis). In continuously varying coordinates this happens at $x = 1$, not $x = 0$. The distance traveled is $vt$, but it also equals 1, so we have

$$t = \frac{1}{v}\,.$$

Again we can say that from the stationary point of view, elapsed times satisfy $t' = t/\gamma$, and from the moving point of view they must satisfy $t = t'/\gamma + \epsilon$ for some gap $\epsilon$, although the geometrical origin of the gap may not be obvious yet. So by the same algebra as in the Lowry case, $\epsilon = v^2 t$. But in the present case that implies

$$\epsilon = v.$$

How can we understand this result? Follow the $x'$ axis ($t' = 0$ curve) around the cylinder; it arrives back at the $t$ axis at $t = \delta$. In stationary coordinates the distance "traveled" by this superluminal path is 1, but its "speed" is $1/v$.

Therefore, $1 = \delta/v$, or

$$\delta = v = \epsilon.$$

Thus $\epsilon$ is the spacing (in $t$, not $t'$) of the helical winding of the $x'$ axis. This shows that the gap term in the moving observer's calculation of the total time of his trip in the stationary observer's clock comes from jumping from one labeling of some $t' = \text{const}$ curve to the next (from $t'$ to $t' + \gamma\epsilon$).

Another way of looking at it is to use (5) of the Brans–Stewart paper, specialized to $n = -1$. This is the claim that the coordinates

$$(x', t') \quad \text{and} \quad (x' - \gamma, t' + \gamma v)$$

represent the same event (space-time point). We can check this from the Lorentz transformation (inverted from (4) of Brans–Stewart)

$$\begin{aligned} x =& \gamma(x' + vt'), \\ t =& \gamma(t' + vx') : \end{aligned}$$

we get

$$x_{\text{new}} = \gamma(x' - \gamma + v(t' + \gamma v)) = \gamma^2(v^2 - 1) + \gamma(x' + vt') = x - 1,$$

$$t_{\text{new}} = \gamma(t' + \gamma v + v(x' - \gamma)) = \gamma(t' + vx') = t\,;$$

but $x$ and $x - 1$ are equivalent, since $x$ is periodic.

Now, if as we agreed

$$t' = \frac{t}{\gamma}\,, \tag{$*$}$$

then

$$t' + \gamma\epsilon = \gamma\left(\frac{t'}{\gamma} + \epsilon\right),$$

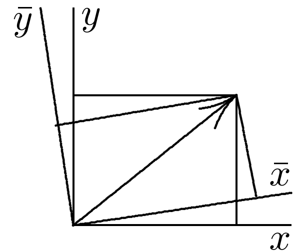
or (since we also agreed $t = t'/\gamma + \epsilon$))

$$t' + \gamma\epsilon = \gamma t. \tag{\#}$$

Comparing $(*)$ and $(\#)$, we see that the $\gamma$ has "flipped" exactly as needed to make the time dilation formula consistent for each observer, provided we insert a gap term $\gamma\epsilon$.

## Vectors *(Chapter 2)*

One must distinguish between vectors as “geometrical objects”, independent of coordinate system, and their *coordinate (component) representations* with respect to some particular coordinate system. Each is valuable in its proper place.




There are two prototypes of vectors in space-time:

- the displacement of one point from another:

$$(\Delta t, \Delta x, \Delta y, \Delta z) \qquad (\Delta t \equiv t_2 - t_1\,, \text{etc.}).$$

- the tangent vector to a curve:

$$\left\{ \frac{dx^\mu(s)}{ds} \right\}$$

(where $s$ is any parameter).

The second is a limiting case of the first (in the sense that derivatives are limiting cases of finite differences). In curved space the first becomes problematical (points can't be subtracted), so the second becomes the leading prototype.

Both of these are so-called *contravariant* vectors. The other type of vectors, *covariant vectors* or *covectors*, will have as their prototype the gradient of a function:

$$\left\{ \frac{\partial \phi}{\partial x^\mu} \right\}.$$

**Notation:** The *summation convention* applies to a pair of repeated indices, one up and one down:

$$\Lambda^\alpha{}_\beta v^\beta \equiv \sum_{\beta=0}^{3} \Lambda^\alpha{}_\beta v^\beta.$$

This is used also for bases:

$$\vec{v} = v^\alpha \vec{e}_\alpha\,.$$

Schutz uses arrows for 4-dimensional vectors and boldface for 3-dimensional ones. Later, covectors will be indicated with a tilde:

$$\tilde{\omega} = \omega_\alpha \tilde{E}^\alpha.$$

**Basis changes vs. coordinate transformations:** Suppose we have two bases, $\{e_\alpha\}$ and $\{d_{\overline{\alpha}}\}$.

$$\vec{v} = v^\alpha \vec{e}_\alpha = v^{\overline{\beta}} \vec{d}_{\overline{\beta}}\,.$$

Then

$$v^{\overline{\beta}} = \Lambda^{\overline{\beta}}{}_{\alpha} v^{\alpha} \iff \vec{e}_{\alpha} = \Lambda^{\overline{\beta}}{}_{\alpha} \vec{d}_{\overline{\beta}}\,.$$

Thus the coordinates and the bases transform *contragrediently* to each other: $\Lambda$ vs. $(\Lambda^{\mathrm{t}})^{-1}$. Later we will find that covector coordinates behave like contravariant basis vectors and vice versa.

**4-velocity:** Mathematically, this is the normalized tangent vector to a timelike curve:

$$U^{\mu} = \frac{\frac{dx^{\mu}}{ds}}{\sqrt{\left|\left(\frac{d\vec{x}}{ds}\right)^{2}\right|}}$$

where $s$ is any parameter. We can introduce *proper time* by

$$d\tau \equiv \sqrt{\left|\left(\frac{d\vec{x}}{ds}\right)^{2}\right|}\, ds;$$

then

$$U^\mu = \frac{dx^\mu}{d\tau} \,.$$

Proper time is the Lorentzian analogue of *arc length* in Euclidean geometry.

The ordinary physical velocity of the particle (3-velocity) is

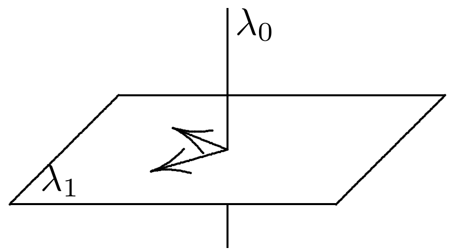
$$\mathbf{v} = \frac{\mathbf{U}}{U^0} \,.$$

Thus

$$U^0 = \frac{1}{\sqrt{1-\mathbf{v}^2}} = \gamma \equiv \cosh\theta, \qquad U^j = \frac{v^j}{\sqrt{1-\mathbf{v}^2}} = \hat{\mathbf{v}}\sinh\theta \quad (\mathbf{U} = \gamma\mathbf{v}).$$

In the particle's rest frame, $\vec{U} = \vec{e}_0$ .

As Schutz hints in Sec. 2.3, "frame" can mean just a split into space and time, rather than a complete basis $\{\vec{e}_0, \vec{e}_1, \vec{e}_2, \vec{e}_3\}$. A frame in this sense is determined by $\vec{U}$ or by $\mathbf{v}$. Different bases compatible with a frame are related by rotations. Compare the eigenspaces of a matrix with the characteristic polynomial $(\lambda - \lambda_0)(\lambda - \lambda_1)^3$; compare the directions in space that are important to a house builder, with or without a street laid out.




**4-momentum:** With a physical particle (or, indeed, a compound physical system) is associated a *rest mass*, $m$, and a 4-vector

$$\vec{p} = m\vec{U}.$$

Then $\vec{p}^2 = -m^2$ (the square being with respect to the Lorentz inner product); and in the rest frame, $p^0 = m$. In another Lorentz frame,

$$p^0 = \frac{m}{\sqrt{1-\mathbf{v}^2}} = m + \frac{1}{2}m\mathbf{v}^2 + \cdots$$

(the *energy*),

$$p^j = \frac{mv^j}{\sqrt{1-\mathbf{v}^2}}$$

(the *momentum*, which is $\gamma$ times the nonrelativistic 3-momentum, $m\mathbf{v}$).

Why is momentum defined this way? As Rindler says (*Essential Relativity*, p. 77),

> If Newton's well-tested theory is to hold in the "slow-motion limit," and unnecessary complications are to be avoided, then only *one* Lorentz-invariant

mechanics appears to be possible. Moreover, it is persuasively elegant, and far simpler that any conceivable alternative.

That is,

1. Kinetic energy and ordinary 3-momentum are important quantities; the 4-momentum construction puts them together into a geometrical object (that is, their values in different reference frames are related so as to make the $p^\mu$ transform like the components of a vector).

2. 4-momentum is conserved (e.g., in collisions).

$$\sum_{i=1}^{N_{\text{initial}}} \vec{p}_i = \sum_{i=1}^{N_{\text{final}}} \vec{p}\,'_i$$

This conservation law

i) is Lorentz-covariant;

ii) reduces to the Newtonian momentum and energy conservation laws when the velocities are small.

It is a *postulate*, verified by experiment.

If time permits, I shall return to Rindler's argument for the inevitability of the form of the 4-momentum.

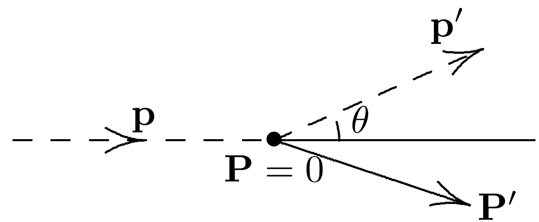
**Photons** travel along null lines, so they have $\vec{p}^2 = 0$. Therefore, for some constant $\hbar\omega$ we must have

$$\vec{p} = \hbar\omega(1, \hat{n}), \qquad |\hat{n}| = 1.$$

A photon has no rest frame. Recall that a null vector or null line is perpendicular to itself in the Lorentz scalar product!

## The Compton effect *(Exercise 32)*

This is a famous application of vectorial thinking.




The (historically important) problem is to find the relation between $\omega'$ and $\theta$.

Incoming electron: $\vec{P} = (m, 0)$.

Incoming photon: $\vec{p} = \hbar\omega(1, \hat{n})$.

Outgoing electron: $\quad \vec{P}\,' = ?.$

Outgoing photon: $\quad \vec{p}\,' = \hbar\omega'(1, \hat{n}').$

These equations are at our disposal:

$$\vec{P} + \vec{p} = \vec{P}\,' + \vec{p}\,',$$

$$(\vec{P}\,')^2 = \vec{P}^2 = -m^2, \qquad \vec{p}^{\,2} = (\vec{p}\,')^2 = 0.$$

Thus (arrows omitted for speed)

$$(P')^2 = [P + (p - p')]^2 = P^2 + 2P \cdot (p - p') + (p - p')^2$$

implies

$$0 = 2P \cdot (p - p') - 2p \cdot p'.$$

Substitute the coordinate expressions for the vectors, and divide by $2\hbar$:

$$m(\omega - \omega') = \hbar\omega\omega'(1 - \hat{n} \cdot \hat{n}').$$

Divide by $m$ and the frequencies to get the difference of wavelengths (divided by $2\pi$):

$$\frac{1}{\omega'} - \frac{1}{\omega} = \frac{\hbar}{m}(1 - \cos\theta).$$

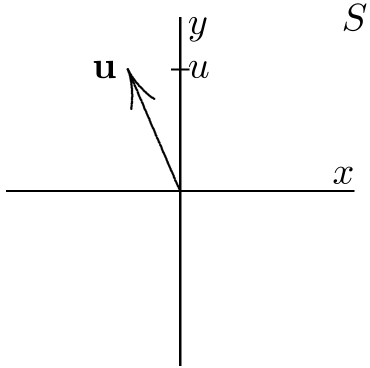
($\hbar/m$ is called the *Compton wavelength* of the electron.) This calculation is generally considered to be much simpler and more elegant than balancing the momenta in the center-of-mass frame and then performing a Lorentz transformation back to the lab frame.

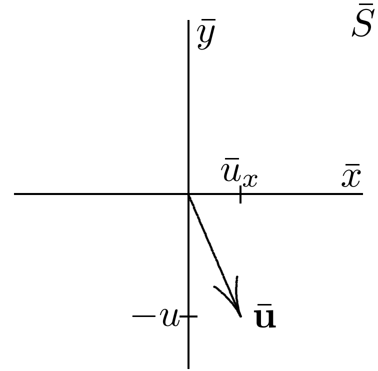
## Inevitability of $\mathbf{p} = \gamma m \mathbf{v}$

I follow Rindler, *Essential Relativity*, Secs. 5.3–4.

Assume that *some* generalization of Newtonian 3-momentum is conserved. By symmetry it must have the form $\mathbf{p} = \mathcal{M}(\mathbf{v})\mathbf{v}$. We want to argue that $\mathcal{M} = m\gamma$.

Consider a glancing collision of two identical particles, A and B, with respective initial inertial frames $S$ and $\overline{S}$. $\overline{S}$ moves with respect to $S$ at velocity $\mathbf{v}$ in the positive $x$ direction. After the collision, each has a transverse velocity component in its own old frame. (Say that that of $S$ is in the positive $y$ direction.) From the symmetry of the situation it seems safe to assume that these transverse velocities are equal and opposite; call their magnitude $u$.




From the relativistic velocity addition law (see below), we find that the transverse velocity of B *relative to S* is

$$\frac{u}{\gamma(v)(1+\mathbf{u}_x v)}\,.$$

The assumed transverse momentum conservation in $S$ thus implies

$$\mathcal{M}(\mathbf{u})u = \mathcal{M}\big(\overline{\mathbf{u}}\big|_S\big)\,\frac{u}{\gamma(1+\mathbf{u}_x v)}\,.$$

In the limit of a glancing collision, $u$ and $\overline{u}_x$ approach 0, and hence $\mathbf{u} \to 0$, $\overline{\mathbf{u}}\big|_S \to \mathbf{v}$. Thus

$$\frac{\mathcal{M}(\mathbf{v})}{\gamma(v)} = \mathcal{M}(\mathbf{u}) \to m,$$

Q.E.D.

### Velocity addition law

Following Rindler Sec. 2.15, let's examine how a velocity $\mathbf{u}$ with respect to a frame $S$ transforms when we switch to a frame $\overline{S}$ moving with velocity $\mathbf{v}$ relative to $S$. Recall:

1. In nonrelativistic kinematics, $\mathbf{u} = \mathbf{v} + \overline{\mathbf{u}}$.

2. If the motion is all in one dimension,

$$u = \frac{v + \overline{u}}{1 + v\overline{u}},$$

corresponding to addition of the rapidities (inverse hyperbolic tangents of the velocities). Our formula must generalize both of these.

By definition,

$$\mathbf{u} = \lim \left( \frac{\Delta x}{\Delta t}, \frac{\Delta y}{\Delta t}, \frac{\Delta z}{\Delta t} \right),$$

$$\overline{\mathbf{u}} = \lim \left( \frac{\Delta \overline{x}}{\Delta \overline{t}}, \frac{\Delta \overline{y}}{\Delta \overline{t}}, \frac{\Delta \overline{z}}{\Delta \overline{t}} \right),$$

Apply the Lorentz transformation, $\Delta\overline{x} = \gamma(\Delta x - v\Delta t)$, etc.:

$$\overline{\mathbf{u}} = \left( \frac{u_x - v}{1 - vu_x}, \frac{u_y}{\gamma(1 - vu_x)}, \frac{u_z}{\gamma(1 - vu_x)} \right).$$

The standard form of the formula is the inverse of this:

$$\mathbf{u} = \left( \frac{\overline{u}_x + v}{1 + v\overline{u}_x}, \frac{\overline{u}_y}{\gamma(1 + v\overline{u}_x)}, \frac{\overline{u}_z}{\gamma(1 + v\overline{u}_x)} \right).$$

The transverse part of this result was used in the momentum conservation discussion above.

Note that the formula is *not symmetric* under interchange of $\mathbf{v}$ and $\mathbf{u}$. The two results differ by a *rotation*. This is the same phenomenon discussed in Schutz, Exercise 2.13, and a handout of mine on "Composition of Lorentz transformations".

Tensors *(Chapter 3)*

## I. Covectors

Consider a linear coordinate transformation,

$$x^{\overline{\alpha}} = \Lambda^{\overline{\alpha}}{}_{\beta} x^{\beta}.$$

Recall that $\{x^{\overline{\alpha}}\}$ and $\{x^{\beta}\}$ label the *same* point $\vec{x}$ in $\mathbf{R}^4$ with respect to different bases. Note that

$$\frac{\partial x^{\overline{\alpha}}}{\partial x^{\beta}} = \Lambda^{\overline{\alpha}}{}_{\beta}\,.$$

Tangent vectors to curves have components that transform just like the coordinates of $\vec{x}$: By the chain rule,

$$v^{\overline{\mu}} \equiv \frac{dx^{\overline{\mu}}}{ds} = \frac{\partial x^{\overline{\mu}}}{\partial x^{\nu}} \frac{dx^{\nu}}{ds} = \Lambda^{\overline{\mu}}{}_{\nu} v^{\nu}.$$

**Interjection** (essential to understanding Schutz's notation): In this chapter Schutz assumes that $\Lambda$ is a Lorentz boost transformation, $\{\Lambda^{\overline{\alpha}}{}_{\beta}\} \overset{\mathcal{O}}{\leftarrow} \Lambda(\mathbf{v})$. (This is unnecessary, in the sense that the tensor concepts being developed apply to *any* linear coordinate transformation.) The mapping in the inverse direction is $\Lambda(-\mathbf{v})$, and it is therefore natural to write for its matrix elements $\Lambda(-\mathbf{v}) = \Lambda^{\gamma}{}_{\overline{\delta}}$, counting on the location of the barred indices to distinguish the two transformations. Unfortunately, this means that in this book you often see a $\Lambda$ where most linear algebra textbooks would put a $\Lambda^{-1}$.

The components of the differential of a function (i.e., its gradient with respect to the coordinates) transform differently from a tangent vector:

$$df \equiv \frac{\partial f}{\partial x^{\mu}}\, dx^{\mu} \equiv \partial_{\mu} f\, dx^{\mu}$$

(with the summation convention in force);

$$\partial_{\overline{\mu}} f = \frac{\partial f}{\partial x^{\overline{\mu}}} = \frac{\partial f}{\partial x^{\nu}}\, \frac{\partial x^{\nu}}{\partial x^{\overline{\mu}}} = \Lambda^{\nu}{}_{\overline{\mu}}\, \partial_{\nu} f.$$

The transformation is the contragredient of that for tangent vectors. (The transpose is explicit; the inverse is implied by the index positions, as just discussed.)

These two transformation laws mesh together nicely in the derivative of a scalar function along a curve:

$$\frac{df}{ds} = \frac{\partial f}{\partial x^\mu}\,\frac{dx^\mu}{ds} = (\partial_\mu f)\,v^\mu . \tag{1}$$

(Here it is understood that we evaluate $\partial_\mu f$ at some $\vec{x}_0$ and evaluate $v^\mu$ at the value of $s$ such that $\vec{x}(s) = \vec{x}_0$ .) It must equally be true that

$$\frac{df}{ds} = (\partial_{\overline{\nu}} f)\,v^{\overline{\nu}} . \tag{2}$$

The two mutually contragredient transformation laws are exactly what's needed to make the chain-rule transformation matrices cancel out, so that (1) and (2) are consistent.

Moreover, (1) says that $\{\partial_\mu f\}$ is the $1 \times n$ matrix of the linear function $\vec{v} \mapsto \frac{df}{ds}$ $(\mathbf{R}^4 \to \mathbf{R})$, $\vec{v}$ itself being represented by a $n \times 1$ matrix (column vector). This brings us to the modern definition of a covector:

**Definition:** For any given vector space $\mathcal{V}$, the linear functions $\tilde{\omega}\colon \mathcal{V} \to \mathbf{R}$ are called *linear functionals* or *covectors*, and the space of all of them is the *dual space*, $\mathcal{V}^*$.

**Definition:** If $\mathcal{V}$ ( $\cong \mathbf{R}^4$) is the space of tangent vectors to curves, then the elements of $\mathcal{V}^*$ are called *cotangent vectors*. Also, elements of $\mathcal{V}$ are called *contravariant vectors* and elements of $\mathcal{V}^*$ are called *covariant vectors*.

**Definition:** A *one-form* is a covector-valued function (a covector field). Thus, for instance, $\partial_\mu f$ as a function of $x$ is a one-form, $\tilde{\omega}$. (More precisely, $\tilde{\omega} \overset{\mathcal{O}}{\to} \{\partial_\mu f\}$.)

**Observation:** Let $\{\omega_\mu\}$ be the matrix of a covector $\tilde\omega$:

$$\tilde\omega(\vec v) = \omega_\mu v^\mu. \tag{3}$$

Then under a change of basis in $\mathcal{V}$ inducing the coordinate change

$$v^{\overline{\alpha}} = \Lambda^{\overline{\alpha}}{}_\beta\, v^\beta,$$

the coordinates (components) of $\tilde\omega$ transform contragrediently:

$$\omega_{\overline{\alpha}} = \Lambda^\beta{}_{\overline{\alpha}}\, \omega_\beta\,.$$

(This is proved by observing in (3) the same cancellation as in (1)–(2).)

Note the superficial resemblance to the transformation law of the basis vectors themselves:

$$\vec e_{\overline{\alpha}} = \Lambda^\beta{}_{\overline{\alpha}}\, \vec e_\beta\,. \tag{4}$$

(Recall that the same algebra of cancellation assures that $\vec{v} = v^{\overline{\alpha}}\vec{e}_{\overline{\alpha}} = v^{\beta}\vec{e}_{\beta}$.) This is the origin of the term "covariant vectors": such vectors transform *along with* the basis vectors of $\mathcal{V}$ instead of contragrediently to them, as the vectors in $\mathcal{V}$ itself do. However, at the less superficial level there are two important differences between (3) and (4):

1. (4) is a relation among vectors, not numbers.

2. (4) relates *different* geometrical objects, not different coordinate representations of the same object, as (3) does.

Indeed, the terminology "covariant" and "contravariant" is nowadays regarded as defective and obsolescent; nevertheless, I often find it useful.

Symmetry suggests the existence of bases for $\mathcal{V}^*$ satisfying

$$\tilde{E}^{\overline{\alpha}} = \Lambda^{\overline{\alpha}}{}_{\beta}\tilde{E}^{\beta}.$$

Sure enough, . . .

> **Definition:** Given a basis $\{\vec{e}_\mu\}$ for $\mathcal{V}$, the *dual basis* for $\mathcal{V}^*$ is defined by
> $$\tilde{E}^\mu(\vec{e}_\nu) = \delta^\mu_\nu \,.$$

In other words, $\tilde{E}^\mu$ is the linear functional whose matrix in the unbarred coordinate system is $(0, 0, \ldots, 1, 0, \ldots)$ with the 1 in the $\mu$th place, just as $\vec{e}_\nu$ is the vector whose matrix is

$$\begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{pmatrix} .$$

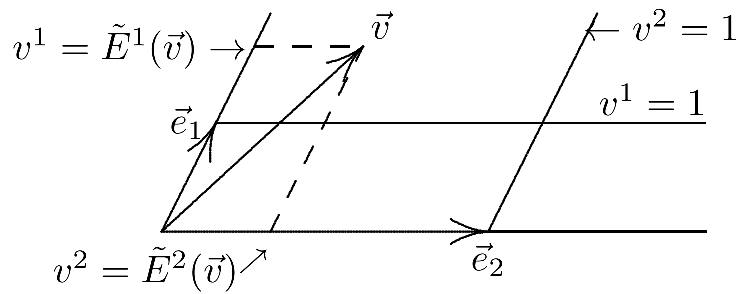
In still other words, $\tilde{E}^\mu$ is the covector that calculates the $\mu$th coordinate of the vector it acts on:

$$\vec{v} = v^\nu \vec{e}_\nu \iff \tilde{E}^\mu(\vec{v}) = v^\mu .$$

Conversely,

$$\tilde{\omega} = \omega_\nu \tilde{E}^\nu \iff \omega_\nu = \tilde{\omega}(\vec{e}_\nu).$$

Note that to determine $\tilde{E}^2$ (for instance), we need to know not only $\vec{e}_2$ but also the other $\vec{e}_\mu$ :




In this two-dimensional example, $\tilde{E}^2(\vec{v})$ is the part of $\vec{v}$ in the direction of $\vec{e}_2$ — *projected along* $\vec{e}_1$ . (As long as we consider only orthogonal bases in Euclidean space and Lorentz frames in flat space-time, this remark is not relevant.)

So far, the metric (indefinite inner product) has been irrelevant to the discussion — except for remarks like the previous sentence. However, if we have a metric, we can use it to *identify* covectors with ordinary vectors. Classically, this is called “raising and lowering indices”. Let us look at this correspondence in three different ways:

**Abstract (algebraic) version:** Given $\vec{u} \in \mathcal{V}$, it determines a $\tilde{\omega} \in \mathcal{V}^*$ by

$$\tilde{\omega}(\vec{v}) \equiv \vec{u} \cdot \vec{v}. \tag{$*$}$$

Conversely, given $\tilde{\omega} \in \mathcal{V}^*$, there is a unique $\vec{u} \in \mathcal{V}$ such that $(*)$ holds. (I won't stop to prove the existence and uniqueness, since they will soon become obvious from the other two versions.)

**Calculational version:** Let's write out $(*)$:

$$\tilde{\omega}(\vec{v}) = -u^0 v^0 + u^1 v^1 + u^2 v^2 + u^3 v^3.$$

Thus

$$\tilde{\omega} \xrightarrow{\mathcal{O}} (-u^0, u^1, u^2, u^3),$$

or

$$\omega_\alpha = \eta_{\alpha\beta} u^\beta.$$

(Here we are assuming an orthonormal basis. Hence the only linear coordinate transformations allowed are Lorentz transformations.) Conversely, given $\tilde{\omega}$ with matrix $\{\omega_\mu\}$, the corresponding $\vec{u}$ has components

$$\begin{pmatrix} -\omega_0 \\ \omega_1 \\ \vdots \end{pmatrix},$$

or $u^\alpha = \eta^{\alpha\beta}\omega_\beta$. (Recall that $\eta$ in an orthonormal basis is numerically its own inverse.)

**Geometrical version:** (For simplicity I describe this in language appropriate to Euclidean space (positive definite metric), not space-time.) $\tilde{\omega}$ is represented by a set of parallel, equally spaced surfaces of *codimension* 1 — that is, dimension $n-1$ ( $=3$ in space-time). These are the *level surfaces* of the linear function $f(\vec{x})$ such that $\omega_\mu = \partial_\mu f$ (a constant covector field). (If we identify points in space with vectors, then $f(\vec{x})$ is the same thing as $\tilde{\omega}(\vec{x})$.) See the drawing on p. 64. Note that a *large* $\tilde{\omega}$ corresponds to *closely spaced* surfaces. If $\vec{v}$ is the displacement between two points $\vec{x}$, then

$$\tilde{\omega}(\vec{v}) \equiv \tilde{\omega}(\Delta\vec{x}) = \Delta f$$

$=$ number of surfaces pierced by $\vec{v}$. Now $\vec{u}$ is the vector normal to the surfaces, with length inversely proportional to their spacing. (It is essentially what is called $\nabla f$ in vector calculus. However, "gradient" becomes ambiguous when nonorthonormal bases are used. Please be satisfied today with a warning without an explanation.)

To justify this picture we need the following fact:

**Lemma:** If $\tilde{\omega} \in \mathcal{V}^*$ is not the zero functional, then the set of $\vec{v} \in \mathcal{V}$ such that $\tilde{\omega}(\vec{v}) = 0$ has codimension 1. (Thus if the space is 3-dimensional, the level surfaces are planes, for example.)

PROOF: This is a special case of a fundamental theorem of linear algebra:

$$\dim \ker + \dim \operatorname{ran} = \dim \operatorname{dom}.$$

Since the range of $\tilde{\omega}$ is a subspace of $\mathbf{R}$ that is not just the zero vector, it has dimension 1. Therefore, the kernel of $\tilde{\omega}$ has dimension $n - 1$.

## II. General tensors

We have met these kinds of objects so far:

$\binom{1}{0}$ Tangent vectors, $\vec{v} \in \mathcal{V}$.

$$v^{\overline{\beta}} = \Lambda^{\overline{\beta}}{}_{\alpha} v^{\alpha} = \frac{\partial x^{\overline{\beta}}}{\partial x^{\alpha}}\, v^{\alpha}.$$

$\binom{0}{1}$ Covectors, $\tilde{\omega} \in \mathcal{V}^*$; $\quad \tilde{\omega} \colon \mathcal{V} \to \mathbf{R}$.

$$\omega_{\overline{\beta}} = \frac{\partial x^{\alpha}}{\partial x^{\overline{\beta}}}\, \omega_{\alpha}\, .$$

**Interjection:** $\mathcal{V}$ may be regarded as the space of linear functionals on $\mathcal{V}^*$: $\vec{v} \colon \mathcal{V}^* \to \mathbf{R}$. In the pairing or contraction of a vector and a covector, $\tilde{\omega}(\vec{v}) = \omega_{\alpha} v^{\alpha}$, either may be thought of as acting on the other.

$\binom{0}{0}$ Scalars, $\mathbf{R}$ (independent of frame).

$\binom{1}{1}$ Operators, $\underset{\sim}{A}\colon \mathcal{V} \to \mathcal{V}$. Such a linear operator is represented by a square matrix:

$$(\underset{\sim}{A}\vec{v})^\alpha = A^\alpha{}_\beta v^\beta.$$

Under a change of frame (basis change), the matrix changes by a *similarity transformation*:

$$A \mapsto \Lambda A \Lambda^{-1}; \qquad A^{\overline{\gamma}}{}_{\overline{\delta}} = \frac{\partial x^{\overline{\gamma}}}{\partial x^\alpha} A^\alpha{}_\beta \frac{\partial x^\beta}{\partial x^{\overline{\delta}}}\,.$$

Thus the row index behaves like a tangent-vector index and the column index behaves like a covector index. This should not be a surprise, because the role (*raison d'être*) of the column index is to "absorb" the components of the input vector, while the role of the row index is to give birth to the output vector.

$\binom{0}{2}$ Bilinear forms, $\underset{\sim}{Q}\colon \mathcal{V} \times \mathcal{V} \to \mathbf{R}$. The metric tensor $\eta$ is an example of such a beast. A more elementary example is the matrix of a conic section:

$$\begin{aligned} \underset{\sim}{Q}(\vec{x}, \vec{x}) &= Q_{\alpha\beta} x^\alpha x^\beta \\ &= 4x^2 - 2xy + y^2 \quad \text{(for example).} \end{aligned}$$

Here *both* indices are designed to absorb an input vector, and hence both are written as subscripts, and both acquire a transformation matrix of the "co" type under a basis change (a "rotation of axes", in the language of elementary analytic geometry):

$$Q \mapsto \left(\Lambda^{-1}\right)^{\mathrm{t}} Q \Lambda^{-1}; \qquad Q_{\overline{\gamma}\,\overline{\delta}} = \frac{\partial x^\alpha}{\partial x^{\overline{\gamma}}} \frac{\partial x^\beta}{\partial x^{\overline{\delta}}} Q_{\alpha\beta} \,.$$

(When both input vectors are the same, the bilinear form is called a *quadratic form*, $\underset{\sim}{Q}\colon \mathcal{V} \to \mathbf{R}$ (nonlinear).)

**Remarks:**

1. In Euclidean space, if we stick to *orthonormal* bases (related by rotations), there is no difference between the operator transformation law and the bilinear form one (because a rotation equals its own contragredient).

2. The metric $\eta$ has a special property: Its components don't change at all if we stick to Lorentz transformations.

**Warning:** A bilinear or quadratic form is not the same as a "two-form". The matrix of a two-form (should you someday encounter one) is *antisymmetric*. The matrix $Q$ of a quadratic form is (by convention) *symmetric*. The matrix of a generic bilinear form has no special symmetry.

**Observation:** A bilinear form can be regarded as a linear mapping from $\mathcal{V}$ into $\mathcal{V}^*$ (since supplying the second vector argument then produces a scalar). Similarly, since $\mathcal{V} = \mathcal{V}^{**}$, a linear operator can be thought of as another kind of bilinear form, one of the type

$$\underset{\sim}{A}: \mathcal{V}^* \times \mathcal{V} \to \mathbf{R}.$$

The second part of this observation generalizes to the official definition of a tensor:

GENERAL DEFINITION OF TENSORS

1. A tensor of type $\binom{0}{N}$ is a real-valued function of $N$ vector arguments,

$$(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_N) \mapsto T(\vec{v}_1, \ldots, \vec{v}_N),$$

   which is linear in each argument when the others are held fixed (*multilinear*). For example,

$$T(\vec{u}, (5\vec{v}_1 + \vec{v}_2), \vec{w}) = 5T(\vec{u}, \vec{v}_1, \vec{w}) + T(\vec{u}, \vec{v}_2, \vec{w}).$$

2. A tensor of type $\binom{M}{N}$ is a real-valued multilinear function of $M$ covectors and $N$ vectors,
$$T(\tilde{\omega}_1, \ldots, \tilde{\omega}_M, \vec{v}_1, \ldots, \vec{v}_N).$$

The *components* (a.k.a. *coordinates*, *coefficients*) of a tensor are equal to its values on a basis (and its dual basis, in the case of a tensor of mixed type):

case $\binom{1}{2}$ :
$$T^{\mu}_{\nu\rho} \equiv T(\tilde{E}^{\mu}, \vec{e}_{\nu}, \vec{e}_{\rho}).$$

Equivalently, the components constitute the matrix by which the action of $T$ is calculated in terms of the components of its arguments (input vectors and covectors):
$$T(\tilde{\omega}, \vec{v}, \vec{u}) = T^{\mu}_{\nu\rho}\omega_{\mu}v^{\nu}u^{\rho}.$$

It follows that under a change of frame the components of $T$ transform by acquiring a transformation matrix attached to each index, of the contravariant or the covariant type depending on the position of the index:

$$T^{\overline{\alpha}}_{\overline{\beta}\overline{\gamma}} = \frac{\partial x^{\overline{\alpha}}}{\partial x^\mu} \frac{\partial x^\nu}{\partial x^{\overline{\beta}}} \frac{\partial x^\rho}{\partial x^{\overline{\gamma}}} T^\mu_{\nu\rho} \,.$$

Any tensor index can be raised or lowered by the metric; for example,

$$T_{\mu\nu\rho} = \eta_{\mu\sigma} T^\sigma_{\nu\rho} \,.$$

Therefore, in relativity, where we always have a metric, the mixed (and the totally contravariant) tensors are not really separate objects from the covariant tensors, $\binom{0}{N}$. In *Euclidean* space with only orthonormal bases, the numerical components of tensors don't even change when indices are raised or lowered! (This is the reason why the entire distinction between contravariant and covariant vectors or indices can be totally ignored in undergraduate linear algebra and physics courses.)

In applications in physics, differential geometry, etc., tensors sometimes arise in their roles as multilinear functionals. (See, for instance, equation (4.14) defining the stress-energy-momentum tensor in terms of its action on two auxiliary vectors.) After all, only scalars have an invariant meaning, so ultimately any tensor in physics ought to appear together with other things that join with it to make an invariant number. However, those "other things" don't have to be individual vectors and covectors. Several tensors may go together to make up a scalar quantity, as in

$$R^{\alpha\beta\gamma\delta} A_{\alpha\beta} A_{\gamma\delta} \,.$$

In such a context the concept and the notation of tensors as multilinear functionals fades into the background, and the tensor component transformation law, which guarantees that the quantity is indeed a scalar, is more pertinent. In olden times, tensors were simply *defined* as lists of numbers (generalized matrices) that transformed in a certain way under changes of coordinate system, but that way of thinking is definitely out of fashion today (even in physics departments).

# On the relation between inversion and index swapping

In special relativity, Schutz writes $\{\Lambda^{\beta}{}_{\bar{\alpha}}\}$ for the matrix of the coordinate transformation inverse to the coordinate transformation

$$x^{\bar{\alpha}} = \Lambda^{\bar{\alpha}}{}_{\beta}\, x^{\beta}. \tag{$*$}$$

However, one might want to use that same notation for the transpose of the matrix obtained by raising and lowering the indices of the matrix in $(*)$:

$$\Lambda_{\bar{\alpha}}{}^{\beta} = g_{\bar{\alpha}\bar{\mu}}\Lambda^{\bar{\mu}}{}_{\nu}g^{\nu\beta}.$$

Here $\{g_{\alpha\beta}\}$ and $\{g_{\bar{\alpha}\bar{\beta}}\}$ are the matrices of the metric of Minkowski space with respect to the unbarred and barred coordinate system, respectively. (The coordinate transformation $(*)$ is linear, but not necessarily a Lorentz transformation.) Let us investigate whether these two interpretations of the symbol $\Lambda^{\beta}{}_{\bar{\alpha}}$ are consistent.

If the answer is *yes*, then (according to the first definition) $\delta^{\bar\alpha}_{\bar\gamma}$ must equal

$$\begin{aligned}
\Lambda^{\bar\alpha}{}_\beta \Lambda_{\bar\gamma}{}^\beta &\equiv \Lambda^{\bar\alpha}{}_\beta \left(g_{\bar\gamma\bar\mu}\Lambda^{\bar\mu}{}_\nu g^{\nu\beta}\right) \\
&= g_{\bar\gamma\bar\mu}\left(\Lambda^{\bar\mu}{}_\nu g^{\nu\beta}\Lambda^{\bar\alpha}{}_\beta\right) \\
&= g_{\bar\gamma\bar\mu} g^{\bar\mu\bar\alpha} \\
&= \delta^{\bar\alpha}_{\bar\gamma}\,, \qquad \text{Q.E.D.}
\end{aligned}$$

(The first step uses the second definition, and the next-to-last step uses the transformation law of a $\binom{2}{0}$ tensor.)

In less ambiguous notation, what we have proved is that

$$\left(\Lambda^{-1}\right)^\beta{}_{\bar\alpha} = g_{\bar\alpha\bar\mu}\Lambda^{\bar\mu}{}_\nu g^{\nu\beta}. \qquad (\dagger)$$

Note that if $\Lambda$ is not a Lorentz transformation, then the barred and unbarred $g$

matrices are not numerically equal; at most one of them in that case has the form

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

If $\Lambda$ *is* Lorentz (so that the $g$ matrices are the same) and the coordinates are with respect to an orthogonal basis(so that indeed $g = \eta$), then (†) is the indefinite-metric counterpart of the "inverse = transpose" characterization of an orthogonal matrix in Euclidean space: *The inverse of a Lorentz transformation equals the transpose with the indices raised and lowered (by $\eta$).* (In the Euclidean case, $\eta$ is replaced by $\delta$ and hence (†) reduces to

$$\left(\Lambda^{-1}\right)^{\beta}{}_{\bar\alpha} = \Lambda^{\bar\alpha}{}_{\beta}\,,$$

in which the up-down index position has no significance.) For a general linear transformation, (†) may appear to offer a free lunch: How can we calculate an inverse matrix without the hard work of evaluating Cramer's rule, or performing a Gaussian elimination? The answer is that in the general case at least one of the matrices $\left\{g_{\bar{\alpha}\bar{\mu}}\right\}$ and $\left\{g^{\nu\beta}\right\}$ is nontrivial and somehow contains the information about the inverse matrix.

*Alternative argument:* We can use the metric to map between vectors and covectors. Since

$$v^{\bar{\alpha}} = \Lambda^{\bar{\alpha}}{}_{\beta} v^{\beta}$$

is the transformation law for vectors, that for covectors must be

$$\begin{aligned}
\tilde{v}_{\bar{\mu}} &= g_{\bar{\mu}\bar{\alpha}} v^{\bar{\alpha}} \\
&= g_{\bar{\mu}\bar{\alpha}} \Lambda^{\bar{\alpha}}{}_{\beta} v^{\beta} \\
&= g_{\bar{\mu}\bar{\alpha}} \Lambda^{\bar{\alpha}}{}_{\beta} g^{\beta\nu} \tilde{v}_{\nu} \\
&\equiv \Lambda_{\bar{\mu}}{}^{\nu} \tilde{v}_{\nu}
\end{aligned}$$

according to the second definition. But the transformation matrix for covectors is the transpose of the inverse of that for vectors — i.e.,

$$\tilde{v}_{\bar{\mu}} = \Lambda^{\nu}{}_{\bar{\mu}} \tilde{v}_{\nu}$$

according to the first definition. Therefore, the definitions are consistent.

## Tensor products

If $\vec{v}$ and $\vec{u}$ are in $\mathcal{V}$, then $\vec{v} \otimes \vec{u}$ is a $\binom{2}{0}$ tensor defined in any of these equivalent ways:

1. $T = \vec{v} \otimes \vec{u}$ has components $T^{\mu\nu} = v^{\mu} u^{\nu}$.

2. $T\colon \mathcal{V}^* \to \mathcal{V}$ is defined by

$$T(\tilde{\omega}) = (\vec{v} \otimes \vec{u})(\tilde{\omega}) \equiv \tilde{\omega}(\vec{u})\, \vec{v}.$$

3. $T\colon \mathcal{V}^* \times \mathcal{V}^* \to \mathbf{R}$ is a bilinear mapping, defined by absorbing two covector arguments:

$$T(\tilde{\omega}, \tilde{\xi}) \equiv \tilde{\omega}(\vec{v})\tilde{\xi}(\vec{u}).$$

4. Making use of the inner product, we can write for any $\vec{w} \in \mathcal{V}$,

$$(\vec{v} \otimes \vec{u})(\vec{w}) \equiv (\vec{u} \cdot \vec{w})\,\vec{v}.$$

(Students of quantum mechanics may recognize the Hilbert-space analogue of this construction under the notation $|v\rangle\langle u|$.)

The tensor product is also called *outer product.* (That's why the scalar product is called "inner".) The tensor product is itself bilinear in its factors:

$$(\vec{v}_1 + z\vec{v}_2) \otimes \vec{u} = \vec{v}_1 \otimes \vec{u} + z\,\vec{v}_2 \otimes \vec{u}.$$

We can do similar things with other kinds of tensors. For instance, $\vec{v} \otimes \tilde{\omega}$ is a $\binom{1}{1}$ tensor (an operator $T\colon \mathcal{V} \to \mathcal{V}$) with defining equation

$$(\vec{v} \otimes \tilde{\omega})(\vec{u}) \equiv \tilde{\omega}(\vec{u})\, \vec{v}.$$

(One can argue that this is an even closer analogue of the quantum-mechanical $|v\rangle\langle\tilde{\omega}|$.)

**A standard basis for each tensor space:** Let $\{\vec{e}_\mu\} \equiv \mathcal{O}$ be a basis for $\mathcal{V}$. Then $\{\vec{e}_\mu \otimes \vec{e}_\nu\}$ is a basis for the $\binom{2}{0}$ tensors:

$$T = T^{\mu\nu} \vec{e}_\mu \otimes \vec{e}_\nu \quad \Longleftrightarrow \quad T \overset{\mathcal{O}}{\to} \{T^{\mu\nu}\} \quad \Longleftrightarrow \quad T^{\mu\nu} = T(\tilde{E}^\mu, \tilde{E}^\nu).$$

Obviously we can do the same for the higher ranks of tensors. Similarly, if $\underset{\sim}{A}$ is a $\binom{1}{1}$ tensor, then

$$\underset{\sim}{A} = A^\mu{}_\nu \, \vec{e}_\mu \otimes \tilde{E}^\nu .$$

Each $\vec{e}_\mu \otimes \tilde{E}^\nu$ is represented by an "elementary matrix" like

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

with the 1 in the $\mu$th row and $\nu$th column.

The matrix of $\vec{v} \otimes \tilde{\omega}$ itself is of the type

$$\begin{pmatrix} v^1\omega_1 & v^1\omega_2 & v^1\omega_3 \\ v^2\omega_1 & v^2\omega_2 & v^2\omega_3 \\ v^3\omega_1 & v^3\omega_2 & v^3\omega_3 \end{pmatrix}.$$

You can quickly check that this implements the operation $(\vec{v} \otimes \tilde{\omega})(\vec{u}) = \tilde{\omega}(\vec{u})\,\vec{v}$. Similarly, $\vec{v} \otimes \vec{u}$ has a matrix whose elements are the products of the components of the two vectors, though the column index is no longer "down" in that case.

We have seen that every $\binom{2}{0}$ tensor is a linear combination of tensor products: $T = T^{\mu\nu}\vec{e}_\mu \otimes \vec{e}_\nu$ . In general, of course, this expansion has more than one term. Even when it does, the tensor may *factor* into a single tensor product of vectors that are not members of the basis:

$$T = (\vec{e}_1 + \vec{e}_2) \otimes (2\vec{e}_1 - 3\vec{e}_2),$$

for instance. (You can use bilinearity to convert this to a linear combination of the standard basis vectors, or vice versa.) However, it is not true that every tensor factors in this way:

$$T = (\vec{e}_0 \otimes \vec{e}_1) + (\vec{e}_2 \otimes \vec{e}_3),$$

for example. (Indeed, if an operator factors, then it has *rank* 1; this makes it a rather special case. Recall that a rank-1 operator has a *one-dimensional range*; the range of $\vec{v} \otimes \vec{u}$ comprises the scalar multiples of $\vec{v}$. This meaning of “rank” is completely different from the previous one referring to the number of indices of the tensor.)

*Symmetries* of tensors are very important. Be sure to read the introductory discussion on pp. 67–68.

## Differentiation of tensor fields (in flat space)

Consider a parametrized curve, $x^\nu(\tau)$. We can define the derivative of a tensor $T(\vec{x})$ along the curve by

$$\frac{dT}{d\tau} = \lim_{\Delta\tau\to 0} \frac{T(\tau+\Delta\tau) - T(\tau)}{\Delta\tau}\,,$$

where $T(\tau)$ is really shorthand for $T(\vec{x}(\tau))$. (In curved space this will need modification, because the tensors at two different points in space can't be subtracted without further ado.) If we use a fixed basis (the same basis vectors for the tangent space at *every* point), then the derivative affects only the components:

$$\frac{dT}{d\tau} = \left(\frac{dT^{\alpha\beta}}{d\tau}\right) \vec{e}_\alpha \otimes \vec{e}_\beta \, .$$

If $\vec{U}$ is the tangent vector to the curve, then

$$\frac{dT^{\alpha\beta}}{d\tau} = \frac{\partial T^{\alpha\beta}}{\partial x^\gamma}\,\frac{dx^\gamma}{d\tau} \equiv T^{\alpha\beta}{}_{,\gamma}\, U^\gamma .$$

The components $\{T^{\alpha\beta}{}_{,\gamma}\}$ make up a $\binom{2}{1}$ tensor, the gradient of $T$:

$$\nabla T = T^{\alpha\beta}{}_{,\gamma}\, \vec{e}_\alpha \otimes \vec{e}_\beta \otimes \tilde{E}^\gamma .$$

Thus

$$T^{\alpha\beta}{}_{,\gamma}\, U^\gamma \overset{\mathcal{O}}{\longleftarrow} \frac{dT}{d\tau} \equiv \nabla T(\vec{U}) \equiv \nabla_{\vec{U}} T .$$

Also, the inner product makes possible the convenient notation

$$\frac{dT}{d\tau} \equiv \vec{U} \cdot \nabla T.$$

# Stress Tensors *(Chapter 4)*

This will be a very quick tour of the most important parts of Chapter 4.

The stress tensor in relativistic physics is also called *energy-momentum tensor.*

The central point of general relativity is that matter is the *source* of gravity, just as charge is the source of electromagnetism. Because gravity is described in the theory by tensors (metric and curvature), the source needs to be a (two-index) tensor. (E&M is a vector theory, and its source 4-vector is built of the charge and the current 3-vector.)

The relation of the stress tensor to more conventional physical quantities is

$$\begin{aligned}
T^{00} &= \rho = \text{energy density}, \\
T^{0i} &= \text{energy flux}, \\
T^{i0} &= \text{momentum density}, \\
&\qquad \text{(actually, } T^{i0} = T^{0i} \text{ in most theories)}, \\
T^{ij} &= \text{momentum flux} = \text{stress}, \\
&\qquad \text{in particular,} \\
T^{ii} &= p = \text{pressure}.
\end{aligned}$$

In terms of the tensor as a bilinear functional, we have (Schutz (4.14))

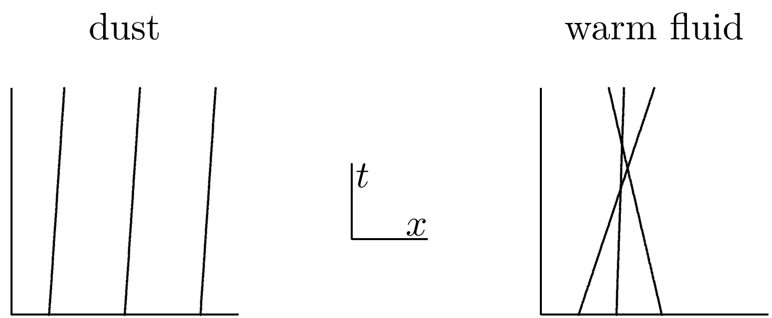
$$\begin{aligned}
T^{\alpha\beta} &= \mathbf{T}(\tilde{E}^\alpha, \tilde{E}^\beta) \equiv \mathbf{T}(\tilde{d}x^\alpha, \tilde{d}x^\beta) \\
&= \text{flux of } \alpha\text{-momentum across a surface of constant } \beta,
\end{aligned}$$

with 0-momentum interpreted as energy.

In standard vector-calculus terms,

$$\tilde{d}t = dx^1\, dx^2\, dx^3 = \tilde{n}\, dS \qquad \text{(for example).}$$

Now consider a cloud of particles all moving at the same velocity. There is a rest Lorentz frame where the speed is 0, and the temperature of this *dust* is 0.



More generally, the temperature will be positive and the particles moving in random directions. (This includes photons as an extreme case.) Even in this case there is a *momentarily comoving rest frame* (MCRF) for the *average* motion inside a small space-time element.

Schutz says that in the MCRF, $T^{0j}$ may still be nonzero in a time-dependent situation, because of heat conduction. The MCRF is not defined by diagonalizing $T^{\alpha\beta}$, but by the physical requirement that the *particles* have no total momentum.

**The conservation law:** $T^{\alpha\beta}{}_{,\beta} \equiv \partial_\beta T^{\alpha\beta} = 0.$

Using Gauss's theorem, this can be integrated to give conservation of total energy and total momentum.

*A hierarchy of matter sources (general to special)*

1. Generic $(T^{\beta\alpha} = T^{\alpha\beta};\ T^{\alpha\beta}{}_{,\beta} = 0)$

2. Fluid (no rigidity $\Rightarrow$ $T^{ij}$ small if $i \neq j$)

3. Perfect fluid (no viscosity; no heat conduction in MCRF)

4. Dust (massive particles; zero temperature)

For perfect fluid, $\mathbf{T}$ is diagonal in a MCRF, and all pressures are equal:

$$\mathbf{T} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \equiv \mathrm{diag}(\rho, p, p, p).$$

In any frame,

$$\mathbf{T} = (\rho + p)\,\vec{U} \otimes \vec{U} + p\,\mathbf{g}^{-1}$$

— because $\mathbf{g}^{-1} = \operatorname{diag}(-1, 1, 1, 1)$ in any Lorentz frame, and $\vec{U} \otimes \vec{U} = \operatorname{diag}(1, 0, 0, 0)$ in MCRF.

In the dust case, $p = 0$ and $\mathbf{T} = \vec{p} \otimes \vec{N} = mn\,\vec{U} \otimes \vec{U}$. Here

$$\vec{p} = m\vec{U} = \text{particle momentum} \quad (m = \text{mass}),$$
$$\vec{N} = n\vec{U} = \text{particle number flux} \quad (n = \text{number density}).$$

## On the Relation of Gravitation to Curvature *(Section 5.1)*

Gravitation forces major modifications to special relativity. Schutz presents the following argument to show that, so to speak, a rest frame is not really a rest frame:

1. Energy conservation (no perpetual motion) implies that photons possess gravitational potential energy: $E' \approx (1 - gh)E$.

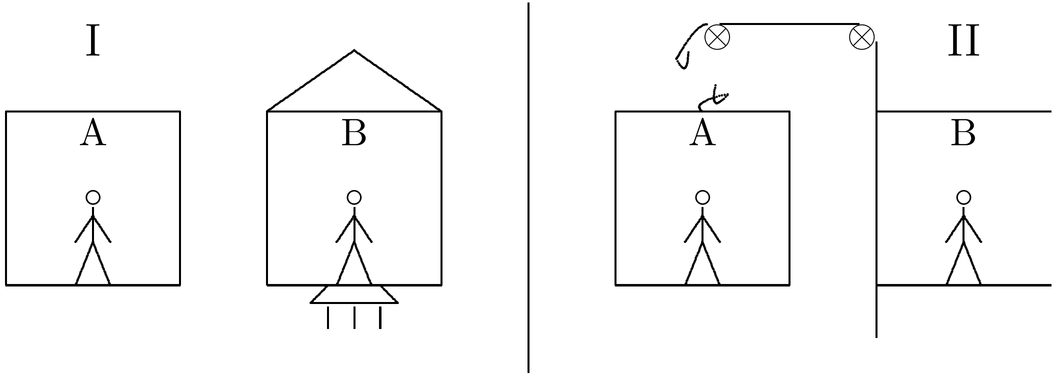
2. $E = h\nu$ implies that photons climbing in a gravitational field are redshifted.

3. Time-translation invariance of photon trajectories plus the redshift imply that a frame at rest in a gravitational field is not inertial!

As Schutz indicates, at least the first two of these arguments can be traced back to Einstein. However, some historical common sense indicates that neither Einstein nor his readers in 1907–1912 could have *assumed* $E = h\nu$ (quantum theory) in order to *prove* the need for something like general relativity. A. Pais, '*Subtle is the Lord ...* ' (Oxford, 1982), Chapters 9 and 11, recounts what Einstein actually wrote.

**1.** Einstein gave *separate* arguments for the energy change and the frequency change of photons in a gravitational field (or accelerated frame). He did not mention $E = h\nu$, but as Pais says, "it cannot have slipped his mind."

**2. The principle of equivalence.** Consider the two famous scenarios:




I. Observer A is in a space station (freely falling). Observer B is passing by in a rocket ship with the rockets turned on (accelerating). A's frame is inertial; he is weightless. B's frame is accelerated; the floor presses up against his feet as if he has weight.

II. Observer A is in an elevator whose cable has just broken. Observer B is standing on a floor of the building next to the elevator's instantaneous position. A's frame is inertial (freely falling). B's frame is at rest on the earth; he is heavy.

In 1907 Einstein gave an inductive argument: Since gravity affects all bodies equally, these situations are operationally indistinguishable by experiments done inside the labs. A's frame is inertial in both cases, and he is weightless. B cannot distinguish the effect of acceleration from the gravity of the earth.

In 1911 Einstein turned this around into a deductive argument: A theory in which these two scenarios are indistinguishable by internal experiments *explains from first principles* why gravity affects all bodies equally.

**3.** Einstein's argument for the frequency shift is just a modification of the Doppler effect argument on p. 115 of Schutz. Summarizing from Pais: Let a frame $\Sigma$ start coincident with a frame $S_0$ and have acceleration $a$. Let light be emitted at point $x = h$ in $S_0$ with frequency $\nu_2$. The light reaches the origin of $\Sigma$ at time $h$ (plus a correction of order $O(h^2)$), when $\Sigma$ has velocity $ah$. It therefore has frequency $\nu_1 = \nu_2(1 + ah)$ to lowest order (cf. Sec. 2.7). Now identify $\Sigma$ with the "heavy observer" in the previous discussion. Then $a = g$, and $ah = \Phi$, the gravitational potential difference between the emission and detection points. Extrapolated to nonuniform gravitational fields, $\nu_1 = \nu_2(1 + \Phi)$ predicts the redshift of light from dense stars, which is observed!

**4.** As remarked, Einstein wrote *two* papers on these matters, in 1907 and 1911. (Note: Full general relativity did not appear till 1915.) As far as I can tell from Pais, *neither* contains the notorious photon perpetual-motion machine! Both papers are concerned with four overlapping issues:

a) the equivalence principle;

b) the gravitational redshift;

c) the gravitational potential energy of light and of other energy;

d) the bending of light by a gravitational field (leading to a famous observational test in 1919).

**5.** Outline of first paper:

1. Equivalence principle by the inductive argument.

2. Consider a uniformly accelerated frame $\Sigma$. Compare with comoving inertial frames at two times. Conclude that clocks at different points in $\Sigma$ run at different rates. Apply equivalence principle to get different rates at different heights in a gravitational field, and hence a redshift.

3. Conclude that $c$ depends on $x$ in Maxwell's equations. Light bending follows. Also, energy conservation in $\Sigma$ implies that any energy generates an additional position-dependent gravitational energy.

## 6. Outline of second paper:

1. Equivalence principle by the deductive argument.

2. Redshift by the Doppler argument; gravitational energy of light by a similar special-relativity argument. [Note: I think that Pais has misread Einstein at one point. He seems to confuse the man in the space station with the man in the building.]

3. Resolve an apparent discrepancy by accepting the uneven rate of clocks.

4. Hence deduce the nonconstant $c$ and the light bending. (Here Maxwell's equations have been replaced by general physical arguments.)

# Curvilinear Coordinates in Flat Space *(Chapter 5)*

## Random remarks on Sec. 5.2

Most of the material in this section has been covered either in earlier courses or in my lectures on Chapter 3.

**Invertibility and nonvanishing Jacobian.** These conditions (on a coordinate transformation) are closely related but not synonymous. The polar coordinate map on a region such as

$$1 < r < 2, \qquad -\pi < \theta < 2\pi$$

(wrapping around, but avoiding, the origin) has nonvanishing Jacobian everywhere, but it is not one-to-one. The transformation

$$\xi = x, \qquad \eta = y^3$$

is invertible, but its Jacobian vanishes at $y = 0$. (This causes the inverse to be nonsmooth.)

**The distinction between vector and covector components of a gradient, and the components with respect to an ON basis.** The discussions on p. 124 and in Sec. 5.5 finish up something I mentioned briefly before. The gradient of a scalar function is fundamentally a one-form, but it can be converted into a vector field by the metric:

$$(\tilde{d}\phi)_\beta \equiv \phi_{,\beta}\,; \qquad (\vec{d}\phi)^\alpha \equiv g^{\alpha\beta}\phi_{,\beta}\,.$$

For instance, in polar coordinates

$$(\vec{d}\phi)^\theta = \frac{1}{r^2}\,\phi_{,\theta} \qquad (\text{but } (\vec{d}\phi)^r = \phi_{,r}).$$

What classical vector analysis books look at is neither of these, but rather the components with respect to a basis of unit vectors. Refer here to Fig. 5.5, to see how the normalization of the basis vectors (in the $\theta$ direction, at least) that are directly associated with the coordinate system varies with $r$. Thus we have

$$\hat{\theta} = \frac{1}{r}\,\vec{e}_\theta = r\,\tilde{E}^\theta \equiv r\,d\theta,$$

where

$$\vec{e}_\theta = \left\{\frac{dx^\mu}{d\theta}\right\} \quad \text{has norm proportional to } r,$$

$$\tilde{E}^\theta = \left\{\frac{\partial\theta}{\partial x^\mu}\right\} \quad \text{has norm proportional to } \frac{1}{r}\,.$$

Abandoning unit vectors in favor of basis vectors that scale with the coordinates may seem like a step backwards — a retreat to coordinates instead of a machinery adapted to the intrinsic geometry of the situation. However, the standard coordinate bases for vectors and covectors have some advantages:

1. They remain meaningful when there is no metric to define "unit vector".

2. They are calculationally easy to work with; we need not constantly shuffle around the square roots of inner products.

3. If a basis is not orthogonal, scaling its members to unit length does not accomplish much.

In advanced work it is common to use a field of orthonormal bases unrelated to any coordinate system. This makes gravitational theories look like gauge theories. It is sometimes called "Cartan's *repère mobile*" (moving frame). Schutz and I prefer to stick to coordinate bases, at least for purposes of an elementary course.

Curvilinear-coordinate basis vectors depend on position, hence have nonzero derivatives. Therefore, differentiating the *components* of a vector field doesn't produce the components of the derivative, in general! The "true" derivative has the components

$$\frac{\partial \vec{v}}{\partial x^\beta} \xrightarrow{\mathcal{O}} v^\alpha{}_{;\beta} = v^\alpha{}_{,\beta} + v^\mu \Gamma^\alpha{}_{\mu\beta}\,, \qquad (*)$$

where the last term can be read as a matrix, labeled by $\beta$, with indices $\alpha$ and $\mu$, acting on $\vec{v}$. The $\Gamma$ terms are the contribution of the derivatives of the basis vectors:

$$\frac{\partial \vec{e}_\alpha}{\partial x^\beta} = \Gamma^\mu{}_{\alpha\beta}\vec{e}_\mu\,.$$

(From this $(*)$ follows by the product rule.)

Equation $(*)$ is not tensorial, because the index $\beta$ is fixed. However, the numbers $v^\alpha{}_{;\beta}$ are the components of a $\binom{1}{1}$ tensor, $\nabla v$. $(*)$ results upon choosing the contravariant vector argument of $\nabla v$ to be the coordinate basis vector in the $\beta$ direction.

In flat space $(*)$ is derived by demanding that $\nabla v$ be a tensor and that it reduce in Cartesian coordinates to the standard matrix of partial derivatives of $\vec{v}$. In curved space $(*)$ will be a *definition* of covariant differentiation. (Here "covariant" is not meant in distinction from "contravariant", but rather in the sense of "tensorial" or "geometrically intrinsic", as opposed to "coordinate-dependent".) To define a covariant derivative operation, we need a set of quantities $\{\Gamma^\alpha{}_{\beta\gamma}\}$ (*Christoffel symbols*) with suitable properties. Whenever there is a metric tensor in the problem, there is a natural candidate for $\Gamma$, as we'll see.

To define a derivative for one-forms, we use the fact that $\omega_\alpha v^\alpha$ is a scalar — so we know what its derivative is — and we require that the product rule hold:

$$(\omega_\alpha v^\alpha)_{;\beta} \equiv \nabla_\beta(\omega_\alpha v^\alpha) = \omega_{\alpha;\beta} v^\alpha + \omega_\alpha v^\alpha{}_{;\beta}\,.$$

But

$$(\omega_\alpha v^\alpha)_{;\beta} = (\omega_\alpha v^\alpha)_{,\beta} = \omega_{\alpha,\beta} v^\alpha + \omega_\alpha v^\alpha{}_{,\beta} \,.$$

Since

$$v^\alpha{}_{;\beta} = v^\alpha{}_{,\beta} + v^\mu \Gamma^\alpha{}_{\mu\beta} \,,$$

it follows that

$$\omega_{\alpha;\beta} = \omega_{\alpha,\beta} - \omega_\mu \Gamma^\mu{}_{\alpha\beta} \,.$$

These two formulas are easy to remember (given that indices must contract in pairs) if you learn the mnemonic "plUs – Up".

By a similar argument one arrives at a formula for the covariant derivative of any kind of tensor. For example,

$$\nabla_\beta B^\mu{}_\nu = B^\mu{}_{\nu,\beta} + B^\alpha{}_\nu \Gamma^\mu{}_{\alpha\beta} - B^\mu{}_\alpha \Gamma^\alpha{}_{\nu\beta} \,.$$

## Metric compatibility and [lack of] torsion

By specializing the tensor equations to Cartesian coordinates, Schutz verifies in flat space:

(1) $g_{\alpha\beta;\mu} = 0 \quad$ (i.e., $\nabla \mathbf{g} = 0$).

(2) $\Gamma^{\mu}{}_{\alpha\beta} = \Gamma^{\mu}{}_{\beta\alpha}\,.$

(3) $\Gamma^{\mu}{}_{\alpha\beta} = \dfrac{1}{2}\, g^{\mu\gamma}\bigl(g_{\gamma\beta,\alpha} + g_{\alpha\gamma,\beta} - g_{\alpha\beta,\gamma}\bigr).$

**Theorem:** (1) and (2) imply (3), for *any* metric (not necessarily flat). Thus, given a metric tensor (symmetric, invertible), there is a *unique* connection (covariant derivative) that is both *metric-compatible* (1) and *torsion-free* (2). (There are infinitely many other connections that violate one or the other of the two conditions.)

Metric compatibility (1) guarantees that the metric doesn't interfere with differentiation:

$$\nabla_\gamma (g_{\alpha\beta} v^\beta) = g_{\alpha\beta} \nabla_\gamma v^\beta,$$

for instance. I.e., differentiating $\vec{v}$ is equivalent to differentiating the corresponding one-form, $\tilde{v}$.

We will return briefly to the possibility of torsion (nonsymmetric Christoffel symbols) later.

## Transformation properties of the connection

$\Gamma$ is not a tensor! Under a (nonlinear) change of coordinates, it picks up an inhomogeneous term:

$$\Gamma^{\mu'}{}_{\alpha'\beta'} = \frac{\partial x^{\mu'}}{\partial x^\nu} \frac{\partial x^\gamma}{\partial x^{\alpha'}} \frac{\partial x^\delta}{\partial x^{\beta'}} \Gamma^\nu{}_{\gamma\delta} - \frac{\partial x^\gamma}{\partial x^{\alpha'}} \frac{\partial x^\delta}{\partial x^{\beta'}} \frac{\partial^2 x^{\mu'}}{\partial x^\gamma \partial x^\delta} \, .$$

(This formula is seldom used in practice; its main role is just to make the point that the transformation rule is unusual and a mess. We will find better ways to calculate Christoffel symbols in a new coordinate system.) On the other hand,

1. For fixed $\beta$, $\{\Gamma^{\alpha}{}_{\beta\gamma}\}$ is a $\binom{1}{1}$ tensor with respect to the other two indices (namely, the tensor $\nabla\vec{e}_\beta$).

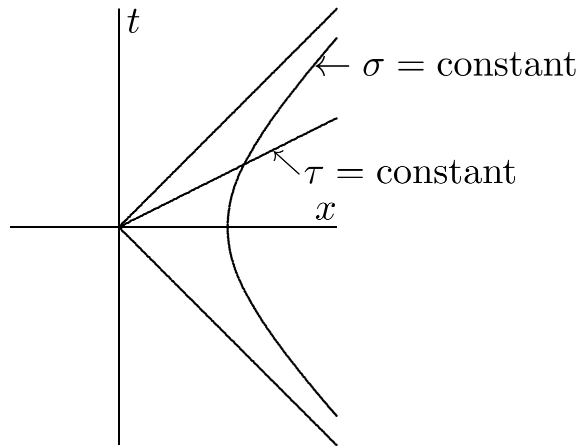
2. $\nabla\vec{v} \overset{\mathcal{O}}{\to} \{\partial_\alpha v^\beta + \Gamma^{\beta}{}_{\mu\alpha}v^\mu\}$ is a $\binom{1}{1}$ tensor, although neither term by itself is a tensor. (Indeed, that's the whole point of covariant differentiation.)

## Tensor Calculus in Hyperbolic Coordinates

We shall do for hyperbolic coordinates in two-dimensional space-time all the things that Schutz does for polar coodinates in two-dimensional Euclidean space.[1]

---

[1] Thanks to Charlie Jessup and Alex Cook for taking notes on my lectures in Fall




2005.

Introduce the coordinates $(\tau, \sigma)$ by

$$t = \sigma \sinh \tau,$$
$$x = \sigma \cosh \tau.$$

Then

$$\frac{t}{x} = \tanh \tau, \qquad -t^2 + x^2 = \sigma^2. \tag{1}$$

The curve $\tau =$ const. is a straight line through the origin. The curve $\sigma =$ const. is a hyperbola. As $\sigma$ varies from 0 to $\infty$ and $\tau$ varies from $-\infty$ to $\infty$ (endpoints not included), the region

$$x > 0, \qquad -x < t < x$$

is covered one-to-one. In some ways $\sigma$ is analogous to $r$ and $\tau$ is analogous to $\theta$, but geometrically there are some important differences.

From Exercises 2.21 and 2.19 we recognize that the hyperbola $\sigma = \text{const.}$ is the path of a uniformly accelerated body with acceleration $1/\sigma$. (The parameter $\tau$ is not the proper time but is proportional to it with a scaling that depends on $\sigma$.)

From Exercises 1.18 and 1.19 we see that translation in $\tau$ (moving the points $(\tau, \sigma)$ to the points $(\tau + \tau_0, \sigma)$) is a Lorentz transformation (with velocity parameter $\tau_0$ ).

Let unprimed indices refer to the inertial coordinates $(t, x)$ and primed indices refer to the hyperbolic coordinates. The equations of small increments are

$$\begin{aligned} \Delta t &= \frac{\partial t}{\partial \tau} \Delta \tau + \frac{\partial t}{\partial \sigma} \Delta \sigma = \sigma \cosh \tau \, \Delta \tau + \sinh \tau \, \Delta \sigma, \\ \Delta x &= \sigma \sinh \tau \, \Delta \tau + \cosh \tau \, \Delta \sigma. \end{aligned} \tag{2}$$

Therefore, the matrix of transformation of (tangent or contravariant) vectors is

$$V^\beta = \Lambda^\beta{}_{\alpha'} V^{\alpha'}, \qquad \Lambda^\beta{}_{\alpha'} = \begin{pmatrix} \sigma\cosh\tau & \sinh\tau \\ \sigma\sinh\tau & \cosh\tau \end{pmatrix}. \tag{3}$$

Inverting this matrix, we have

$$V^{\alpha'} = \Lambda^{\alpha'}{}_\beta V^\beta, \qquad \Lambda^{\alpha'}{}_\beta = \begin{pmatrix} \frac{1}{\sigma}\cosh\tau & -\frac{1}{\sigma}\sinh\tau \\ -\sinh\tau & \cosh\tau \end{pmatrix}. \tag{4}$$

(Alternatively, you could find from (1) the formula for the increments $(\Delta\tau, \Delta\sigma)$ in terms of $(\Delta t, \Delta x)$. But in that case the coefficients would initially come out in terms of the inertial coordinates, not the hyperbolic ones. These formulas would be analogous to (5.4), while (4) is an instance of (5.8–9).)

If you have the old edition of Schutz, be warned that the material on p. 128 has been greatly improved in the new edition, where it appears on pp. 119–120.

Following p. 122 (new edition) we write the transformation of basis vectors

$$\vec{e}_{\alpha'} = \Lambda^{\beta}{}_{\alpha'} \vec{e}_{\beta} \,,$$

$$\begin{aligned} \vec{e}_{\tau} &= \sigma \cosh\tau \, \vec{e}_t + \sigma \sinh\tau \, \vec{e}_x \,, \\ \vec{e}_{\sigma} &= \sinh\tau \, \vec{e}_t + \cosh\tau \, \vec{e}_x \,; \end{aligned} \tag{5}$$

and the transformation of basis covectors

$$\tilde{E}^{\alpha'} = \Lambda^{\alpha'}{}_{\beta} \tilde{E}^{\beta} \,,$$

which is now written in a new way convenient for coordinate systems,

$$\begin{aligned} \tilde{d}\tau &= \frac{1}{\sigma} \cosh\tau \, \tilde{d}t - \frac{1}{\sigma} \sinh\tau \, \tilde{d}x \,, \\ \tilde{d}\sigma &= -\sinh\tau \, \tilde{d}t + \cosh\tau \, \tilde{d}x \,. \end{aligned} \tag{6}$$

To check that the notation is consistent, note that (because our two $\Lambda$ matrices are inverses of each other)

$$\tilde{d}\xi^{\alpha'}(\vec{e}_{\beta'}) = \delta^{\alpha'}_{\beta'} \equiv \tilde{E}^{\alpha'}(\vec{e}_{\beta'}).$$

Note that equations (6) agree with the "classical" formulas for the differentials of the curvilinear coordinates as scalar functions on the plane; it follows that, for example, $\tilde{d}\tau(\vec{v})$ is (to first order) the change in $\tau$ under a displacement from $\vec{x}$ to $\vec{x}+\vec{v}$. Note also that the analog of (6) in the reverse direction is simply (2) with $\Delta$ replaced by $\tilde{d}$.

## *The metric tensor*

*Method 1:* By definitions (see (5.30))

$$g_{\alpha'\beta'} = \mathsf{g}(\vec{e}_{\alpha'}, \vec{e}_{\beta'}) = \vec{e}_{\alpha'} \cdot \vec{e}_{\beta'}\,.$$

So

$$g_{\tau\tau} = -\sigma^2, \qquad g_{\sigma\sigma} = 1, \qquad g_{\tau\sigma} = g_{\sigma\tau} = 0.$$

These facts are written together as

$$ds^2 = -\sigma^2\, d\tau^2 + d\sigma^2,$$

or

$$\mathsf{g} \overset{\mathcal{O}'}{\to} \begin{pmatrix} -\sigma^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

The inverse matrix, $\{g^{\alpha'\beta'}\}$, is

$$\begin{pmatrix} -\frac{1}{\sigma^2} & 0 \\ 0 & 1 \end{pmatrix}.$$

*Method 2:* In inertial coordinates

$$\mathrm{g} \overset{\mathcal{O}}{\to} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} .$$

Now use the $\binom{0}{2}$ tensor transformation law

$$g_{\alpha'\beta'} = \Lambda^{\gamma}{}_{\alpha'} \Lambda^{\delta}{}_{\beta'} g_{\gamma\delta} \ ,$$

which in matrix notation is

$$\begin{pmatrix} g_{\tau\tau} & g_{\tau\sigma} \\ g_{\sigma\tau} & g_{\sigma\sigma} \end{pmatrix} = \begin{pmatrix} \Lambda^{t}{}_{\tau} & \Lambda^{t}{}_{\sigma} \\ \Lambda^{x}{}_{\tau} & \Lambda^{x}{}_{\sigma} \end{pmatrix}^{\mathrm{t}} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Lambda^{t}{}_{\tau} & \Lambda^{t}{}_{\sigma} \\ \Lambda^{x}{}_{\tau} & \Lambda^{x}{}_{\sigma} , \end{pmatrix}$$

which, with (3), gives the result.

This calculation, while conceptually simple, is cumbersome and subject to error in the index conventions. Fortunately, there is a streamlined, almost automatic, version of it:

*Method 3:* In the equation $ds^2 = -dt^2 + dx^2$, write out the terms via (2) and simplify, treating the differentials as if they were numbers:

$$\begin{aligned} ds^2 &= -(\sigma \cosh\tau\, d\tau + \sinh\tau\, d\sigma)^2 + (\sigma \sinh\tau\, d\tau + \cosh\tau\, d\sigma.)^2 \\ &= -\sigma^2\, d\tau^2 + d\sigma^2. \end{aligned}$$

*Christoffel symbols*

A generic vector field can be written

$$\vec{v} = v^{\alpha'} \vec{e}_{\alpha'}\ .$$

If we want to calculate the derivative of $\vec{v}$ with respect to $\tau$, say, we must take into account that the basis vectors $\{\vec{e}_{\alpha'}\}$ depend on $\tau$. Therefore, the formula for such a derivative in terms of components and coordinates contains extra terms, with coefficients called *Christoffel symbols.* [See $(*)$ and the next equation several pages ago, or (5.43,46,48,50) in the book.]

The following argument shows the most elementary and instructive way of calculating Christoffel symbols for curvilinear coordinates in flat space. Once we get into curved space we won't have inertial coordinates to fall back upon, so other methods of getting Christoffel symbols will need to be developed.

Differentiate (5) to get

$$
\begin{aligned}
\frac{\partial \vec{e}_\tau}{\partial \tau} &= \sigma \sinh \tau \, \vec{e}_t + \sigma \cosh \tau \, \vec{e}_x = \sigma \vec{e}_\sigma \, , \\
\frac{\partial \vec{e}_\tau}{\partial \sigma} &= \cosh \tau \, \vec{e}_t + \sinh \tau \, \vec{e}_x = \frac{1}{\sigma} \, \vec{e}_\tau \, , \\
\frac{\partial \vec{e}_\sigma}{\partial \tau} &= \cosh \tau \, \vec{e}_t + \sinh \tau \, \vec{e}_x = \frac{1}{\sigma} \, \vec{e}_\tau \, , \\
\frac{\partial \vec{e}_\sigma}{\partial \sigma} &= 0.
\end{aligned}
$$

Since by definition

$$\frac{\partial \vec{e}_{\alpha'}}{\partial x^{\beta'}} = \Gamma^{\mu'}_{\alpha'\beta'}\vec{e}_{\mu'} \ ,$$

we can read off the Christoffel symbols for the coordinate system $(\tau, \sigma)$:

$$\Gamma^{\sigma}_{\tau\tau} = \sigma, \quad \Gamma^{\tau}_{\tau\tau} = 0,$$

$$\Gamma^{\tau}_{\tau\sigma} = \Gamma^{\tau}_{\sigma\tau} = \frac{1}{\sigma} \ ,$$

$$\Gamma^{\sigma}_{\tau\sigma} = \Gamma^{\sigma}_{\sigma\tau} = 0,$$

$$\Gamma^{\tau}_{\sigma\sigma} = 0, \quad \Gamma^{\sigma}_{\sigma\sigma} = 0.$$

Later we will see that the Christoffel symbol is necessarily *symmetric in its subscripts*, so in dimension $d$ the number of independent Christoffel symbols is

$$d \text{ (superscripts)} \times \frac{d(d+1)}{2} \text{ (symmetric subscript pairs)} = \tfrac{1}{2}d^2(d+1).$$

For $d = 2, 3, 4$ we get $6, 18, 40$ respectively. In particular cases there will be geometrical symmetries that make other coefficients equal, make some of them zero, etc.

# Manifolds and Curvature *(Chapter 6)*

## Random remarks on Secs. 6.1–3

My lectures on Chap. 6 will fall into two parts. First, I assume (as usual) that you are reading the book, and I supply a few clarifying remarks. In studying this chapter you should pay close attention to the valuable summaries on pp. 143 and 165.

Second, I will talk in more depth about selected topics where I feel I have something special to say. Some of these may be postponed until after we discuss Chapters 7 and 8, so that you won't be delayed in your reading.

**Manifolds.** In essence, an $n$-dimensional manifold is a set in which a point can be specified by $n$ numbers (coordinates). We require that *locally* the manifold "looks like" $\mathbf{R}^n$ in the sense that any function on the manifold is continuous, differentiable, etc. if and only if it has the corresponding property as a function of the coordinates. (Technically, we require that any two coordinate systems are related to each other in their region of overlap (if any) by a smooth (infinitely differentiable) function, and then we *define* a function on the manifold to be, for instance, once differentiable if it is once differentiable as a function of any (hence every) coordinate set.) This is a *weaker* property than the statement that the manifold locally "looks like" *Euclidean* $n$-dimensional space. That requires not only a differentiable structure, but also a metric to define angles and distances. (In my opinion, Schutz's example of a cone is an example of a nonsmooth Riemannian metric, or of a nonsmooth embedding into a higher-dimensional space, not of a nonsmooth manifold.)

*Globally*, the topology of the manifold may be different from that of $\mathbf{R}^n$. In practice, this means that no single coordinate chart can cover the whole space. However, frequently one uses coordinates that cover all but a lower-dimensional singular set, and can even be extended to that set in a discontinuous way. An elementary example (taught in grade-school geography) is the sphere. The usual system of latitude and longitude angles is singular at the Poles and necessarily discontinuous along some line joining them (the International Date Line being chosen by convention). Pilots flying near the North Pole use charts based on a local Cartesian grid, not latitude and longitude (since "all directions are South" is a useless statement).

**Donaldson's Theorem.** In the early 1980s it was discovered that $\mathbf{R}^4$ (and no other $\mathbf{R}^n$) admits two inequivalent differentiable structures. Apparently, nobody quite understands intuitively what this means. The proof appeals to gauge field theories. See *Science* **217**, 432–433 (July 1982).

**Metric terminology.** Traditionally, physicists and mathematicians have used different terms to denote metrics of various signature types. Also relevant here is the term used for the type of partial differential equation associated with a metric via $g^{\mu\nu}\partial_\mu\partial_\nu + \cdots$.

| **Physics** | **Math (geometry)** | **PDE** |
|---|---|---|
| Euclidean | Riemannian | elliptic |
| Riemannian | semi-Riemannian | either |
| Riemannian with indefinite metric | Lorentzian (pseudo-Riemannian) | hyperbolic |

In a Lorentzian space Schutz writes

$$g \equiv \det\bigl(g_{\mu\nu}\bigr), \qquad dV = \sqrt{-g}\, d^4x.$$

Some other authors write

$$g \equiv |\det\bigl(g_{\mu\nu}\bigr)|, \qquad dV = \sqrt{g}\, d^4x.$$

**Local Flatness Theorem:** At any one point $\mathcal{P}$, we can choose coordinates so that

$$\mathcal{P} \overset{\mathcal{O}}{\to} \{0,0,0,0\} \quad \text{and} \quad g_{\alpha\beta}(x) = \eta_{\alpha\beta} + O(x^2).$$

That is,

$$g_{\alpha\beta}(\mathcal{P}) = \eta_{\alpha\beta} \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad g_{\alpha\beta,\gamma}(\mathcal{P}) = 0.$$

From the last equation, $\Gamma^{\alpha}{}_{\beta\gamma}(\mathcal{P}) = 0$ follows.

Schutz gives an ugly proof based on Taylor expansions and counting. The key step is that the derivative condition imposes 40 equations, while there are 40 unknowns (degrees of freedom) in the first derivatives of the coordinate transformation we are seeking. Schutz does not check that this square linear system is nonsingular, but by looking closely at (6.26) one can see that its kernel is indeed zero. (Consider separately the four cases: all indices equal, all distinct, $\nu' = \gamma'$, $\nu' = \mu'$.)

I will present a more interesting proof of this theorem later, after we study geodesics.

**More numerology.** Further counting on p. 150 shows that *if* $n = 4$, there are 20 independent true degrees of freedom in the second derivatives of the metric (i.e., in the curvature). Out of curiosity, what happens if $n = 2$ or 3? The key fact used (implicit in the discussion on p. 149) is

$$\left(\begin{array}{l}\text{The number of independent components of a}\\ \text{symmetric } \binom{0}{3} \text{ tensor (or other 3-index quan-}\\ \text{tity) in dimension } n\end{array}\right) = \frac{n(n+1)(n+2)}{3!}\,.$$

The generalization to $p$ symmetric indices is

$$\frac{(n+p-1)!}{(n-1)!p!} = \binom{n+p-1}{p}.$$

(This is the same as the number of ways to put $p$ bosons into $n$ quantum states.)

Proof: A component (of a *symmetric* tensor) is labelled by telling *how many* indices take each value (or how many particles are in each state). So, count all permutations of $p$ things and the $n-1$ dividers that cut them into equivalence classes labelled by the possible values. Then permute the members of each class among themselves and divide, to remove duplications.

Now, it follows that

$$\Lambda^{\alpha'}{}_{\lambda,\mu\nu} = \frac{\partial^3 x^{\alpha'}}{\partial x^\lambda \, \partial x^\mu \, \partial x^\nu}$$

has $n^2(n+1)(n+2)/6$ independent components. Also, $g_{\alpha\beta,\mu\nu}$ has $[n(n+1)/2]^2 = n^2(n+1)^2/4$. The excess of the latter over the former is the number of components in the curvature.

| $n$ | $g$ | $\Lambda$ | $R$ |
|---|---|---|---|
| 1 | 1 | 1 | 0 |
| 2 | 9 | 8 | 1 |
| 3 | 36 | 30 | 6 |
| 4 | 100 | 80 | 20 |
| 5 | 225 | 175 | 50 |

**The Levi-Civita connection.** We define covariant differentiation by the condition that it reduces at a point $\mathcal{P}$ to ordinary differentiation in locally inertial coordinates at $\mathcal{P}$ (i.e., the coordinates constructed in the local flatness theorem). This amounts to saying that the Christoffel symbols, hence $(\vec{e}_\alpha)_{;\beta}$ , vanish at $\mathcal{P}$ in that system. This definition implies

*i*) no torsion ($\Gamma^\gamma{}_{\alpha\beta} = \Gamma^\gamma{}_{\beta\alpha}$);

*ii*) metric compatibility ($\nabla \mathbf{g} = 0$). Therefore, as in flat space, $\Gamma$ is uniquely determined as

$$\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2}\, g^{\mu\gamma}\big(g_{\gamma\beta,\alpha} + g_{\alpha\gamma,\beta} - g_{\alpha\beta,\gamma}\big).$$

Note that other connections, without one or both of these properties, are possible. (Schutz's argument that physics requires ($i$) is not convincing.)

**Integration over surfaces; covariant version of Gauss's theorem.** The notation in (6.43–45) is ambiguous. I understand $n_\alpha$ and $d^3S$ to be the "apparent" unit normal and volume element *in the chart*, so that the classical Gauss theorem can be applied in $\mathbf{R}^4$. The implication is that the combination $n_\alpha \sqrt{-g}\, d^3S$ is independent of chart. (Whenever we introduce a coordinate system into a Riemannian manifold, there are *two* Riemannian geometries in the problem: the "true" one, and the (rather arbitrary) flat one *induced* by the coordinate system, which (gratuitously) identifies part of the manifold with part of a Euclidean space.)

Consider a chart such that $x^0 = 0$ is the 3-surface (not necessarily timelike) and lines of constant $(x^1, x^2, x^3)$ are initially normal to the 3-surface. (All this is *local* and patched together later.) Then

$$\sqrt{-g} = \sqrt{|g_{00}|}\sqrt{{}^{(3)}g},$$

and

$$\mathcal{N}_\alpha \equiv n_\alpha \sqrt{|g_{00}|} = (\sqrt{|g_{00}|}, 0, 0, 0)$$

is a unit normal 1-form in the true geometry (since $g^{0i} = 0$ and $g^{00}(\mathcal{N}_0)^2 = -1$). (For simplicity I ignore the possibility of a null 3-surface.) Thus $n_\alpha \sqrt{-g}\, d^3S = \mathcal{N}_\alpha \sqrt{{}^{(3)}g}\, d^3S$ is an intrinsic geometric object, because $\sqrt{{}^{(3)}g}\, d^3S$ is the Riemannian volume on the 3-surface as a manifold in its own right. (Note that in these coordinates $d^3S = dx^1\, dx^2\, dx^3$.)

Let us relate this discussion to the treatment of surface integrals in traditional vector calculus. There, an "element of surface area", denoted perhaps by $d\sigma$, is used to define integrals of scalar functions and flux integrals of vector fields. (We have now dropped by one dimension from the setting of the previous discussion: The surface in question is 2-dimensional and the embedding space is Euclidean 3-space.) The notion of area of a surface clearly depends on the metric of space, hence, ultimately, on the dot product in $\mathbf{R}^3$. However, I claim that *the flux of a vector field through a surface is independent of the dot product.* Such a flux integral is traditionally expressed in terms of a "vectorial element of surface", $\hat{n}\,d\sigma$, where $\hat{n}$ is a unit vector normal to the surface. Note that both "unit" and "normal" are defined in terms of the dot product! The point is that, nevertheless, $\hat{n}\,d\sigma$ really should be thought of as a metric-independent unit, although the two factors are metric-dependent.

One can show that $d\sigma = \sqrt{{}^{(2)}g}\,d^2S$ in the notation of the previous discussion. Therefore, $\hat{n}$ is the same as the $\mathcal{N}_\alpha$ there, vectors being identified with one-forms via the Euclidean metric in an orthonormal frame, where index-raising is numerically trivial.

To demonstrate the claim, let $(u^1, u^2)$ be parameters on a surface in Euclidean $\mathbf{R}^3$. Then

(1) A vector normal to the surface is $\frac{\partial \vec{x}}{\partial u^1} \times \frac{\partial \vec{x}}{\partial u^2}$ (since the factors are tangent to the surface). One divides by $\left\| \frac{\partial \vec{x}}{\partial u^1} \times \frac{\partial \vec{x}}{\partial u^2} \right\|$ to get a *unit* normal, $\hat{n}$.

(2) The covariant surface area element is

$$d^2\sigma = \sqrt{{}^{(2)}g}\, du^1\, du^2 = \left\| \frac{\partial \vec{x}}{\partial u^1} \times \frac{\partial \vec{x}}{\partial u^2} \right\| du^1\, du^2$$

(the area of an infinitesimal parallelogram).

Therefore, the two normalization factors cancel and one gets

$$\hat{n}\, d^2\sigma = \left( \frac{\partial \vec{x}}{\partial u^1} \times \frac{\partial \vec{x}}{\partial u^2} \right) du^1\, du^2.$$

This is formula makes no reference to the metric (dot product), though $\left\| \frac{\partial \vec{x}}{\partial u^1} \times \frac{\partial \vec{x}}{\partial u^2} \right\|$ *does.* This explains the disappearance of the concept "unit". The disappearance of the concept "normal" from the definition is explained by the replacement of the normal vector $\hat{n}$ by the one-form $\mathcal{N}_\alpha$, which is intrinsically associated with the surface without the mediation of a metric.

More generally, the formalism of *differential forms* cuts through all the metric-dependent and dimension-dependent mess to give a unified theory of integration over submanifolds. The things naturally integrated over $p$-dimensional submanifolds are *$p$-forms.* For example,

$$v^\alpha \, n_\alpha \sqrt{-g} \, d^3S = v^\alpha \, \mathcal{N}_\alpha \sqrt{{}^{(3)}g} \, d^3S$$

is a 3-form constructed out of a vector field in a covariant (chart-independent, "natural") way; its integral over a surface gives a scalar. Chapter 4 of Schutz's gray book gives an excellent introduction to integration of forms.

**Parallel transport.** We say that a vector field $\vec{V}$ defined on a curve is *parallel-transported* through $\mathcal{P}$ if it moves through $\mathcal{P}$ as if instantaneously constant in the local inertial frame. This is as close as we can come to requiring $\vec{V}$ to be "locally constant" — in particular, in curved space we can't require such a condition throughout a whole region, only on individual curves. More precisely, if $\vec{U} \equiv \frac{d\vec{x}}{d\lambda}$ is the tangent vector to the curve, then $\vec{V}$ is parallel-transported along the curve if and only if

$$0 = \frac{d\vec{V}}{d\lambda} \equiv \vec{U} \cdot \nabla \vec{V} \xrightarrow{\mathcal{O}} \{U^\beta V^\alpha{}_{;\beta}\}.$$

In coordinates, this is

$$0 = U^\beta \frac{\partial V^\alpha}{\partial x^\beta} + U^\beta \Gamma^\alpha{}_{\gamma\beta} V^\gamma$$

(where the first term is the ordinary directional derivative of the components, $d(V^\alpha)/d\lambda$). This is a first-order, linear ordinary differential equation that $\vec{V}$ satisfies. Note that only derivatives of $V^\alpha$ along the curve count. So $\vec{U} \cdot \nabla\vec{V} = \nabla_{\vec{U}}\vec{V}$ is defined even if $\vec{V}$ is defined only *on the curve* — although $\nabla\vec{V} \overset{\mathcal{O}}{\to} \{V^\alpha{}_{;\beta}\}$ isn't!

More generally,

$$\frac{d(V^\alpha)}{d\lambda} + \Gamma^\alpha{}_{\gamma\beta} U^\beta V^\gamma \equiv \left(\frac{d\vec{V}}{d\lambda}\right)^\alpha$$

is called the *absolute derivative* of $\vec{V}(\lambda)$, when the latter is a vector-valued function defined on the curve whose tangent is $\vec{U}(\lambda)$. Schutz routinely writes $U^\beta V^\alpha{}_{;\beta}$ for the absolute derivative even when $\vec{V}$ is undefined off the curve (e.g., when $\vec{V}$ is the velocity or momentum of a particle). This can be justified. (If $\vec{V} = \vec{V}(x)$

is the velocity field of a fluid, it's literally OK.) (Many books write $\frac{D\vec{V}}{d\lambda}$, instead of $\frac{d\vec{V}}{d\lambda}$, for the absolute derivative, to emphasize that it's a covariantly defined quantity, not just the collection of derivatives of the component functions.)

**Geodesic equation.** If the tangent vector of a curve is parallel-transported along the curve itself, then the curve is as close as we can come in curved space to a straight line. Written out, this says

$$0 = (\vec{V} \cdot \nabla \vec{V})^\alpha = V^\beta \frac{\partial V^\alpha}{\partial x^\beta} + \Gamma^\alpha{}_{\beta\gamma} V^\beta V^\gamma,$$

or

$$0 = \frac{d^2 x^\alpha}{d\lambda^2} + \Gamma^\alpha{}_{\beta\gamma}\big(x(\lambda)\big) \frac{dx^\beta}{d\lambda} \frac{dx^\gamma}{d\lambda}. \qquad (\dagger)$$

This is a second-order, nonlinear ODE for $x(\lambda)$.

**Reparametrization.** If (and only if) $\phi = f(\lambda)$ and $\phi \neq a\lambda + b$ with $a$ and

$b$ constant, then $x$ as a function of $\phi$ does *not* satisfy the geodesic equation. In what sense is the tangent vector not parallel-transported in this situation?

(ANSWER: Normalization is not constant.)

A "good" reparametrization ($\phi$ *does* equal $a\lambda + b$) is called *affine*.

**Theorem** (cf. Ex. 13)**:** If $x(\lambda)$ is a geodesic (affinely parametrized) and $\vec{V} = \frac{dx}{d\lambda}$, then $\mathbf{g}(\vec{V}, \vec{V}) = g_{\alpha\beta} V^\alpha V^\beta$ is a constant along the curve.

SOFT PROOF: Use the Leibniz rule for $\nabla$, plus $\nabla \mathbf{g} = 0$ and $\vec{V} \cdot \nabla \vec{V} = 0$.

HARD VERIFICATION: Use the Leibniz rule for $\partial$, plus the geodesic equation and the formula for $\Gamma$.

**Length and action.** Now consider any curve (not necessarily a geodesic)

$x\colon [\lambda_0, \lambda_1] \to M$ and its tangent vector $\vec{V} \equiv \frac{dx}{d\lambda}$. (Assume that $x(\lambda)$ is $C^0$ and piecewise $C^1$.) Assume that the curve is not both timelike and spacelike (that is, either $\vec{V} \cdot \vec{V} \geq 0$ for all $\lambda$ or $\vec{V} \cdot \vec{V} \leq 0$ for all $\lambda$).

The *[arc] length* of the curve is

$$s \equiv \int_{\lambda_0}^{\lambda_1} |\vec{V} \cdot \vec{V}|^{1/2} \, d\lambda$$

$$= \int_{\lambda_0}^{\lambda_1} \left| g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} \right|^{1/2} d\lambda$$

(which is independent of the parametrization)

$$\equiv \int_{\text{curve}} |g_{\alpha\beta} \, dx^\alpha \, dx^\beta|^{1/2}$$

$$\equiv \int_{\text{curve}} ds.$$

If the curve is not null, the mapping $\lambda \mapsto s$ is invertible, so $s$ is a special choice of parameter (a new and better $\lambda$). Any parameter related to $s$ by an affine (inhomogeneous linear) transformation, $\phi = as + b$, is called an *affine parameter* for the curve.

The *action* or *energy* of the curve is

$$\begin{aligned}\sigma &\equiv \frac{1}{2}\int_{\lambda_0}^{\lambda_1} \vec{V}\cdot\vec{V}\,d\lambda \\ &= \int_{\lambda_0}^{\lambda_1} \tfrac{1}{2} g_{\alpha\beta}\,\dot{x}^\alpha \dot{x}^\beta\,d\lambda.\end{aligned}$$

Note that the integrand looks like a kinetic energy. This integral is *not* independent of the parametrization.

We can use these integrals like Lagrangians in mechanics and come up with the geodesics as the solutions. We consider variations of the curve, with fixed endpoints and a fixed parametrization interval $[\lambda_0, \lambda_1]$.

**Theorem:**

A) A nonnull, not necessarily affinely parametrized geodesic is a stationary point of the length of the curve: $\delta s = 0$.

B) An affinely parametrized, possibly null geodesic is a stationary point of the action of the curve: $\delta\sigma = 0$. Conversely, a stationary point of $\sigma$ is an affinely parametrized geodesic.

C) For an affinely parametrized geodesic,

$$\begin{aligned}\sigma &= \pm\tfrac{1}{2}(\lambda_1 - \lambda_0)^{-1} s^2 \\ &= \pm\tfrac{1}{2} s^2 \quad \text{if the interval is } [0, 1].\end{aligned}$$

(Note that for a general curve, $\sigma$ may have no relation to $s$.)

Proof of C from B: For an affinely parametrized geodesic, $\vec{V} \cdot \vec{V}$ is a constant, so both integrals can be evaluated:

$$\sigma = \tfrac{1}{2}(\lambda_1 - \lambda_0)\vec{V} \cdot \vec{V}, \qquad s = (\lambda_1 - \lambda_0)|\vec{V} \cdot \vec{V}|^{1/2}.$$

Proof of B: $\delta\sigma = 0$ is equivalent to the Euler–Lagrange equation

$$\frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} - \frac{\partial \mathcal{L}}{\partial x^\alpha} = 0,$$

where

$$\mathcal{L} = \tfrac{1}{2} g_{\alpha\beta}\, \dot{x}^\alpha\, \dot{x}^\beta.$$

Thus

$$\frac{\partial \mathcal{L}}{\partial x^\alpha} = \tfrac{1}{2} g_{\gamma\beta,\alpha}\, \dot{x}^\gamma\, \dot{x}^\beta,$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} = g_{\alpha\beta}\,\dot{x}^\beta,$$

$$\frac{d}{d\lambda}\,\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} = g_{\alpha\beta}\,\ddot{x}^\beta + g_{\alpha\beta,\gamma}\,\dot{x}^\gamma\,\dot{x}^\beta.$$

The equation, therefore, is

$$0 = g_{\alpha\beta}\,\ddot{x}^\beta + \tfrac{1}{2}\big(g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\gamma\beta,\alpha}\big)\,\dot{x}^\beta\,\dot{x}^\gamma,$$

which is the geodesic equation.

PROOF OF A: The Lagrangian $\mathcal{L}$ of this new problem is $\sqrt{\mathcal{L}'}$, where $\mathcal{L}'$ is, up to a factor $\pm 2$, the Lagrangian of the old problem, B. Therefore, we can write

$$\frac{\partial \mathcal{L}}{\partial x^\alpha} = \frac{1}{2\sqrt{\mathcal{L}'}}\,\frac{\partial \mathcal{L}'}{\partial x^\alpha}\,,$$

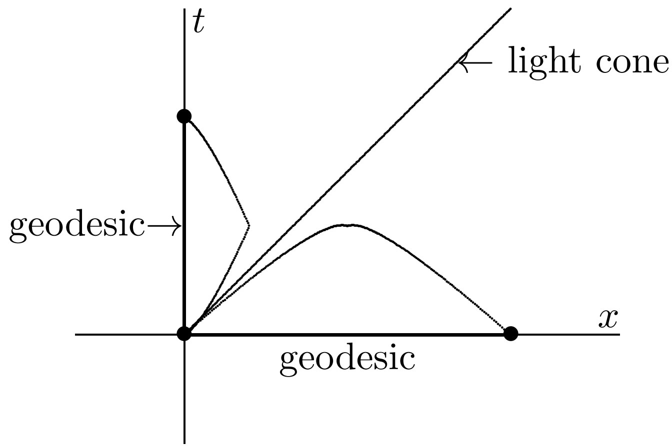
and similarly for the $\dot{x}$ derivative. (The denominator causes no problems, because by assumption $\mathcal{L}' \neq 0$ for the curves under consideration.) Thus the Euler–Lagrange equation is

$$\begin{aligned} 0 &= \frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} - \frac{\partial \mathcal{L}}{\partial x^\alpha} \\ &= \frac{1}{2\sqrt{\mathcal{L}'}}\left[\frac{d}{d\lambda}\frac{\partial \mathcal{L}'}{\partial \dot{x}^\alpha} - \frac{\partial \mathcal{L}'}{\partial x^\alpha}\right] + \frac{1}{2}\frac{\partial \mathcal{L}'}{\partial \dot{x}^\alpha}\frac{d}{d\lambda}\frac{1}{\sqrt{\mathcal{L}'}}\,. \end{aligned}$$

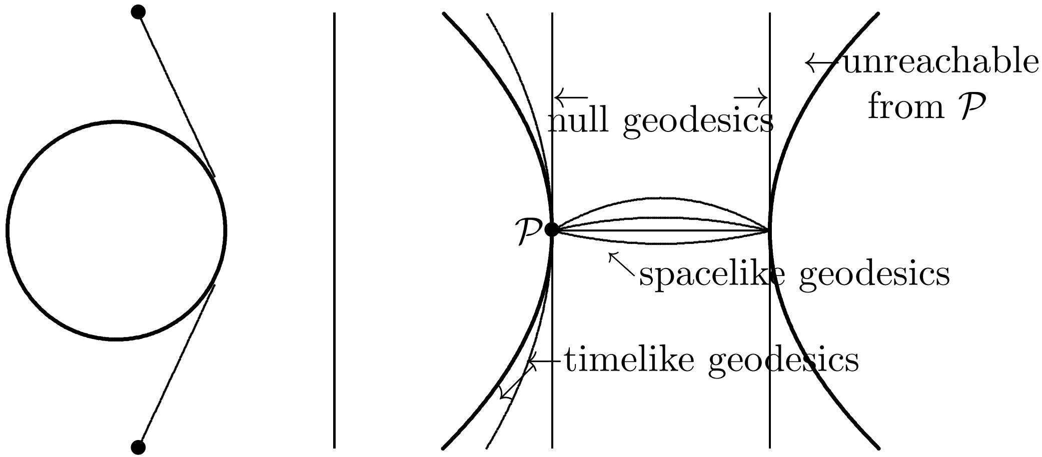
If the curve is an affinely parametrized geodesic, then both of these terms equal 0 and the equation is satisfied. What happens if the curve is not affinely parametrized? Well, we know that $s = \int \mathcal{L}$ is invariant under reparameterizations, so its stationary points must always be the same paths. (Only the labelling of the points by $\lambda$ can change.) Therefore, our differential equation *must* be just the geodesic equation, generalized to arbitrary parametrization. This can be verified by a direct calculation.

**Remarks:**

1. A stationary point is not necessarily a minimum. When the metric is Lorentzian, a timelike geodesic is a local *maximum* of $s$, and a spacelike geodesic is a saddle point. This is intuitively clear from the fact that null lines have zero length:



2. In the Riemannian case, if the manifold is *complete* (every sequence that looks like it ought to have a limit (is *Cauchy*) does have a limit), then any two points are connected by at least one geodesic, namely, the curve that minimizes the distance between them. There may be other geodesics; for example, on a sphere, two typical points are joined by a second geodesic, which maximizes the distance, and antipodal points are joined by infinitely many geodesics, all absolute minimizers. If the Riemannian manifold has holes, a minimizing curve may not exist. A Lorentzian manifold, even if complete, may have pairs of points that are not connected by any geodesics. (*De Sitter space* is a famous example.)




3. The extra symmetry of the $s$ Lagrangian (under nonlinear reparametrization) and the correponding extra looseness of the equations provide a model of *gauge symmetry* and of symmetry under general coordinate transformations in general relativity. Choosing affine parametrization is an example of *gauge fixing*, like Lorenz gauge in electromagnetism. (Choosing $t = x^0$ as

the parameter, as in nonrelativistic physics, is another choice of gauge, like Coulomb gauge in electromagnetism.) This observation is associated with the name of Karel Kuchař.

4. The variational principle $\delta\sigma = 0$ gives a quick way of calculating Christoffel symbols — more efficient than the formula in terms of derivatives of the metric. For example, consider polar coordinates in $\mathbf{R}^2$. We have

$$
\begin{aligned}
\sigma &= \int_{\lambda_0}^{\lambda_1} \frac{1}{2} g_{\alpha\beta}\, \dot{x}^\alpha \dot{x}^\beta \, d\lambda \\
&= \int_{\lambda_0}^{\lambda_1} \frac{1}{2} (\dot{r}^2 + r^2 \dot{\theta}^2) \, d\lambda.
\end{aligned}
$$

Thus the Lagrangian is

$$
\mathcal{L} = \frac{1}{2} (\dot{r}^2 + r^2 \dot{\theta}^2).
$$

The resulting geodesic equations are

$$0 = \frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial \dot{r}} - \frac{\partial \mathcal{L}}{\partial r} = \ddot{r} - r\dot{\theta}^2,$$

$$0 = \frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{d\lambda}(r^2\dot{\theta}) = r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}.$$

But we know that the geodesic equation has the form (†):

$$0 = \ddot{x}^\alpha + \Gamma^\alpha{}_{\beta\gamma}\,\dot{x}^\beta\dot{x}^\gamma.$$

Comparing:

$$\Gamma^r{}_{\theta\theta} = -r, \qquad \Gamma^r{}_{rr} = 0 = \Gamma^r{}_{\theta r}\,,$$

$$\Gamma^\theta{}_{r\theta} = +\frac{1}{r}\,, \qquad \Gamma^\theta{}_{\theta\theta} = 0 = \Gamma^\theta{}_{rr}\,.$$

(Note that the off-diagonal coefficients need to be divided by 2 because they occur in two identical terms in (†).) But these formulas are the same as (5.44).

I shall define curvature by the commutator approach. (Read Schutz for the definition in terms of parallel transport around a closed path.)

Recall that covariant differentiation maps $\binom{p}{q}$-tensors to $\binom{p}{q+1}$-tensors. Therefore, the *second* derivative is a $\binom{p}{q+2}$-tensor. For example,

$$\begin{aligned} V^\alpha{}_{;\mu\nu} &= (V^\alpha{}_{;\mu})_{,\nu} + \Gamma^\alpha{}_{\beta\nu}(V^\beta{}_{;\mu}) - \Gamma^\beta{}_{\mu\nu}V^\alpha{}_{;\beta} \\ &= V^\alpha{}_{,\mu\nu} + (\Gamma^\alpha{}_{\beta\mu}V^\beta)_{,\nu} + \Gamma^\alpha{}_{\beta\nu}(V^\beta{}_{;\mu}) - \Gamma^\beta{}_{\mu\nu}V^\alpha{}_{;\beta} \\ &= V^\alpha{}_{,\mu\nu} + \Gamma^\alpha{}_{\beta\mu,\nu}V^\beta + \Gamma^\alpha{}_{\beta\mu}V^\beta{}_{,\nu} \\ &\quad + \Gamma^\alpha{}_{\beta\nu}V^\beta{}_{,\mu} - \Gamma^\beta{}_{\mu\nu}V^\alpha{}_{,\beta} + \Gamma^\alpha{}_{\beta\nu}\Gamma^\beta{}_{\gamma\mu}V^\gamma - \Gamma^\beta{}_{\mu\nu}\Gamma^\alpha{}_{\gamma\beta}V^\gamma. \end{aligned}$$

Now contemplate (without writing it down) $V^a{}_{;\nu\mu}$. In the foregoing expression, the terms involving derivatives of $V$ components are symmetric, and the others are not. Therefore,

$$V^{\alpha}{}_{;\nu\mu} - V^{\alpha}{}_{;\mu\nu} = (\text{something})^{\alpha}{}_{\beta\mu\nu} V^{\beta}. \tag{$\dagger$}$$

Since $\vec{V}$ is arbitrary, "(something)" is a linear map from $\binom{1}{0}$-tensors to $\binom{1}{2}$-tensors; hence it is a $\binom{1}{3}$-tensor. It is the *Riemann tensor*, $R^{\alpha}{}_{\beta\mu\nu}$. Returning to the calculation, we find that

$$R^{\alpha}{}_{\beta\mu\nu} = -\Gamma^{\alpha}{}_{\beta\mu,\nu} + \Gamma^{\alpha}{}_{\beta\nu,\mu} + \Gamma^{\alpha}{}_{\gamma\mu}\Gamma^{\gamma}{}_{\beta\nu} - \Gamma^{\alpha}{}_{\gamma\nu}\Gamma^{\gamma}{}_{\beta\mu}\,.$$

Recall that

$$V^{\alpha}{}_{;\mu\nu} \equiv \nabla_{\nu}\nabla_{\mu}V^{\alpha}.$$

(Note the reversal of index order.) Thus we can write the fundamental equation as

$$[\nabla_{\mu}, \nabla_{\nu}]V^{\alpha} \equiv (\nabla_{\mu}\nabla_{\nu} - \nabla_{\nu}\nabla_{\mu})V^{\alpha} = R^{\alpha}{}_{\beta\mu\nu}V^{\beta}. \tag{$*$}$$

This is often called the *Ricci identity*. (Note that the "practical person's" version, in terms of subscripts, has the indices arranged in the most practical way: The equation (†) is a machine for replacing indices in abnormal order with indices in standard order.)

For future reference I note that the second covariant derivative of a *scalar* function is independent of index order. This is proved by a calculation similar to the foregoing one for derivatives of vector fields. Both calculations appeal to the symmetry of the Christoffel symbol in its subscripts, hence their conclusions do not apply to connections with torsion. Note also that the *third* derivative of a scalar is the second derivative of a [co]vector, so the order of the outer two indices does matter.

**Alternative version.** Most modern expositions written by mathematicians present the foregoing development in a different way.

From a vector field, $\vec{U}$, we can form a differential operator

$$\nabla_{\vec{U}} = \vec{U} \cdot \nabla \equiv U^\mu \nabla_\mu \, .$$

Let's look first at the commutator of two such operators acting on a *scalar* function (cf. Exercise 6.39(a)):

$$\begin{aligned}
[\vec{U} \cdot \nabla, \vec{W} \cdot \nabla] f &= U^\mu \nabla_\mu (W^\nu \nabla_\nu f) - W^\nu \nabla_\nu (U^\mu \nabla_\mu f) \\
&= U^\mu W^\nu \nabla_\mu \nabla_\nu f - U^\mu W^\nu \nabla_\nu \nabla_\mu f \\
&\quad + U^\mu (\nabla_\mu W^\nu) \nabla_\nu f - W^\nu (\nabla_\nu U^\mu) \nabla_\mu f \\
&= (U^\mu W^\nu{}_{;\mu} - W^\mu U^\nu{}_{;\mu}) \nabla_\nu f
\end{aligned}$$

(since the second derivative of a scalar is symmetric)

$$= (U^\mu W^\nu{}_{,\mu} - W^\mu U^\nu{}_{,\mu}) \nabla_\nu f$$

(since the $\Gamma$ terms cancel)

$$\equiv [\vec{U}, \vec{W}] \cdot \nabla f = \nabla_{[\vec{U}, \vec{W}]} f.$$

We can think of the vector field $\vec{U}$ and the first-order linear partial differential operator $\nabla_{\vec{U}}$ as being in some sense *the same thing*. (Recall that a tangent vector is a way to manufacture a scalar out of the gradient of a scalar function.) Under this identification, the commutator of two vector fields is a new vector field, with components given by either of the last two substantive lines in the calculation above.

With this preliminary out of the way, we can look at the commutator of $\nabla_{\vec{U}}$ and $\nabla_{\vec{W}}$ acting on *vector* fields.

$$\begin{aligned}
[\vec{U}\cdot\nabla, \vec{W}\cdot\nabla]V^\alpha &= U^\mu\nabla_\mu(W^\nu\nabla_\nu V^\alpha) - W^\nu\nabla_\nu(U^\mu\nabla_\mu V^\alpha) \\
&= U^\mu W^\nu\nabla_\mu\nabla_\nu V^\alpha - U^\mu W^\nu\nabla_\nu\nabla_\mu V^\alpha \\
&\quad + U^\mu(\nabla_\mu W^\nu)\nabla_\nu V^\alpha - W^\nu(\nabla_\nu U^\mu)\nabla_\mu V^\alpha \\
&= U^\mu W^\nu[\nabla_\mu, \nabla_\nu]V^\alpha + (U^\mu W^\nu{}_{;\mu} - W^\mu U^\nu{}_{;\mu})\nabla_\nu V^\alpha \\
&= R(\vec{V}, \vec{U}, \vec{W})^\alpha + ([\vec{U}, \vec{W}]\cdot\nabla V)^\alpha .
\end{aligned}$$

In the last step, we have given the first term a new name, and reexpressed the other term in terms of the commutator vector field discovered in the scalar calculation. Since $R(\vec{V},\vec{U},\vec{W})$ depends multilinearly on its three arguments, it defines (or is defined by, or simply *is*) a tensor:

$$R(\vec{V},\vec{U},\vec{W}) \stackrel{\mathcal{O}}{\to} R^{\alpha}{}_{\beta\mu\nu} V^{\beta} U^{\mu} W^{\nu}.$$

Solving our hard-won equation for $R$, we have

$$R(\vec{V},\vec{U},\vec{W}) = [\vec{U}\cdot\nabla, \vec{W}\cdot\nabla]\vec{V} - [\vec{U},\vec{W}]\cdot\nabla V. \qquad (**)$$

$(**)$ can be used as the *definition* of the curvature tensor.

Let us compare $(**)$ with $(*)$. If we think of $\vec{U}$ and $\vec{W}$ as basis vectors in the $\mu$ and $\nu$ directions, then the relation of $(*)$ to the first term in $(**)$ seems clear. Why, then, do we need the second term? The reason is that when $\vec{U}\cdot\nabla$

acts on $\vec{W} \cdot \nabla \vec{V}$, it must hit $\vec{W}$ as well as $\vec{V}$ (and likewise with the vector fields interchanged). We *don't want* these derivatives of the auxiliary vector fields to be in the definition of the curvature tensor, so we have to subtract them off.

The philosophical reason one might prefer the second approach is that it hews closer to the definition of a tensor as a multilinear functional by supplying the auxiliary vector fields as arguments of that functional. One can be confident that ultimately one is always dealing with invariantly defined scalars. (Personally, I don't find this argument persuasive; I think the second derivative of a $\binom{p}{q}$-tensor is perfectly clearly defined as a $\binom{p}{q+2}$-tensor, and I prefer the definition $(*)$, where no correction term is necessary.)

**Warning:** Some people insist on regarding $\nabla_\mu$ as just shorthand for $\vec{e}_\mu \cdot \nabla \equiv (\vec{e}_\mu)^\alpha \nabla_\alpha$ in some basis $\{\vec{e}_\mu(x)\}$ of vector *fields*. From that point of view we ought always to use $(**)$ instead of $(*)$. If this basis is a *coordinate* basis, then $[\vec{e}_\mu, \vec{e}_\nu] = 0$, and hence there is no discrepancy between the two formulas. But for a *noncoordinate basis* the commuatator of the basis fields is generally not zero (see Sec. 5.6). Therefore, for a noncoordinate basis ($*$) is false — *in that interpretation.* I insist, however, that ($*$) is a valid tensor equation when properly understood; it refers to the components of $\binom{1}{2}$-tensors with respect to an arbitrary basis *at a point.*

**Tensors of other types.** ($*$) generalizes to tensors with several indices just as the Christoffel formula for the covariant derivative does: You hang a Riemann tensor on *each* index and add up the terms. For instance,

$$T^{\alpha\beta}{}_{;\nu\mu} = T^{\alpha\beta}{}_{;\mu\nu} + R^{\alpha}{}_{\sigma\mu\nu} T^{\sigma\beta} + R^{\beta}{}_{\sigma\mu\nu} T^{\alpha\sigma} .$$

The plUs–Up rule applies here too; that is, when applied to *covariant* indices the Riemann tensor acquires a minus sign. Thus

$$F^{\mu}{}_{\nu;\beta\alpha} = F^{\mu}{}_{\nu;\alpha\beta} + R^{\mu}{}_{\sigma\alpha\beta} F^{\sigma}{}_{\nu} - R^{\sigma}{}_{\nu\alpha\beta} F^{\mu}{}_{\sigma} .$$

We'll soon see that $R$ is *antisymmetric* in its first two indices; therefore, this equation is equivalent to Schutz's equation (6.78):

$$F^{\mu}{}_{\nu;\beta\alpha} = F^{\mu}{}_{\nu;\alpha\beta} + R^{\mu}{}_{\sigma\alpha\beta}F^{\sigma}{}_{\nu} + R_{\nu}{}^{\sigma}{}_{\alpha\beta}F^{\mu}{}_{\sigma}.$$

It is to that equation that Schutz's parenthetical remark about index-raising applies.

**Symmetries of the Riemann tensor.**

(1) $$R^{\alpha}{}_{\beta\mu\nu} = -R^{\alpha}{}_{\beta\nu\mu}\,.$$

(This is always true, by virtue of the definition of $R$ in terms of a commutator.)

(2) $$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu}\,.$$

(This is proved for the metric-compatible, torsionless connection only. It has been dubbed the "zeroth Bianchi identity".)

(3) $$R^{\alpha}{}_{\mu\nu\rho} + R^{\alpha}{}_{\nu\rho\mu} + R^{\alpha}{}_{\rho\mu\nu} = 0.$$

(This is called the *cyclic identity* or the “first Bianchi identity”.)

(4) $$R^\alpha{}_{\beta\mu\nu;\rho} + R^\alpha{}_{\beta\nu\rho;\mu} + R^\alpha{}_{\beta\rho\mu;\nu} = 0.$$

(This is the *Bianchi identity* — called the “second” by those who also call (3) a Bianchi identity.)

(5) $$R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta}\,.$$

(Obviously (5) (given (1)) implies (2), but it is easier to prove (2) first.)

PROOF OF (2): $0 = g_{\mu\nu;\beta\alpha} - g_{\mu\nu;\alpha\beta} = -R^\gamma{}_{\mu\alpha\beta}g_{\gamma\nu} - R^\gamma{}_{\nu\alpha\beta}g_{\mu\gamma} = -R_{\nu\mu\alpha\beta} - R_{\mu\nu\alpha\beta}\,.$

PROOF OF (3) AND (4): Start from [an instance of] the *Jacobi identity*:

$$[\nabla_\mu,[\nabla_\nu,\nabla_\rho]]V^\alpha + [\nabla_\nu,[\nabla_\rho,\nabla_\mu]]V^\alpha + [\nabla_\rho,[\nabla_\mu,\nabla_\nu]]V^\alpha = 0.$$

(If you write it out, you'll see that all terms cancel in pairs.) Let's look at the first term in detail:

$$\begin{aligned}
&\nabla_\mu(R^\alpha{}_{\beta\nu\rho}V^\beta) - [\nabla_\nu, \nabla_\rho](V^\alpha{}_{;\mu}) \\
&\qquad = R^\alpha{}_{\beta\nu\rho;\mu}V^\beta + R^\alpha{}_{\beta\nu\rho}V^\beta{}_{;\mu} - R^\alpha{}_{\beta\nu\rho}V^\beta{}_{;\mu} + R^\gamma{}_{\mu\nu\rho}V^\alpha{}_{;\gamma} \\
&\qquad = R^\alpha{}_{\beta\nu\rho;\mu}V^\beta + R^\gamma{}_{\mu\nu\rho}V^\alpha{}_{;\gamma}\,.
\end{aligned}$$

Adding the similar contributions from the other two terms, we get, schematically,

$$0 = (4)^\alpha V^\beta + (3)^\gamma V^\alpha{}_{;\gamma}\,.$$

Since the $\vec{V}$ and $\nabla\vec{V}$ at any one point are completely arbitrary, the coefficients (4) and (3) must vanish identically, QED.

PROOF OF (5): Use (1), (2), and (3).

$$R_{\mu\nu\alpha\beta} = -R_{\mu\alpha\beta\nu} - R_{\mu\beta\nu\alpha} = R_{\alpha\mu\beta\nu} - R_{\beta\mu\alpha\nu}\,.$$

Therefore, on the one hand,

$$R_{\mu\nu\alpha\beta} = -R_{\alpha\nu\mu\beta} - R_{\alpha\beta\nu\mu} - R_{\beta\mu\alpha\nu} = R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} - R_{\beta\mu\alpha\nu}\,,$$

but on the other hand,

$$R_{\mu\nu\alpha\beta} = R_{\alpha\mu\beta\nu} + R_{\beta\alpha\nu\mu} + R_{\beta\nu\mu\alpha} = R_{\alpha\beta\mu\nu} + R_{\alpha\mu\beta\nu} - R_{\beta\nu\alpha\mu}\,.$$

Therefore, adding,

$$2(R_{\mu\nu\alpha\beta} - R_{\alpha\beta\mu\nu}) = R_{\alpha\nu\beta\mu} R_{\alpha\mu\beta\nu} - R_{\beta\mu\alpha\nu} - R_{\beta\nu\alpha\mu}\,.$$

The left side of this equation is antisymmetric in $\{\mu\nu\}$ and the right side is symmetric in $\{\mu\nu\}$. Therefore, both of them must be the zero tensor.

# Advanced Topics on Geodesics and Curvature

## Geodesic Deviation

**The physics: Tides.** The equivalence principle notwithstanding, a sufficiently large falling elevator on earth *can* be distinguished from an inertial frame in empty space. A naturally spherical body tends to become a cigar.

**The mathematics.** A geodesic is a map $x(\lambda)$ from $I \subseteq \mathbf{R} \to M$. Consider a whole family of these,

$$x_\epsilon(\lambda) \qquad (-\epsilon_0 < \epsilon < \epsilon_0),$$

obtained by varying the initial data of $x_0 \equiv x$. Then $x_\epsilon(\lambda)$ is a function of two variables, $I \times J \subseteq \mathbf{R}^2 \to M$. We are interested in how neighboring geodesics behave as seen from $x_0$. That is, how does $x$ behave as $\epsilon$ varies with $\lambda$ fixed? To first order, this question is answered by

$$\vec{W} \equiv \frac{\partial x}{\partial \epsilon}(0, \lambda),$$

a vector field. Another vector field in the problem is $\vec{U} \equiv \frac{\partial x}{\partial \lambda}$, the tangent vector to the geodesics.

*The goal: The equation of geodesic deviation*

This is (6.87) in Schutz's book.

$$\frac{D^2 W^\alpha}{d\lambda^2} \equiv \nabla_{\vec{U}} \nabla_{\vec{U}} W^\alpha = R^\alpha{}_{\mu\nu\beta} U^\mu U^\nu W^\beta.$$

(For simplicity I assume no torsion.) The equation is a second-order linear "equation of motion" for $\vec{W}$ along $x_0$. I'll derive it in two ways.

*Classical applied-math approach (perturbation theory)*

The geodesic equation is

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha{}_{\beta\gamma}\big(x(\lambda)\big)\,\frac{dx^\beta}{d\lambda}\,\frac{dx^\gamma}{d\lambda} = 0. \tag{1}$$

Substitute

$$x(\lambda) = x_0(\lambda) + \epsilon W(\lambda) + O(\epsilon^2) \tag{2}$$

where $x_0$ is already a solution of (1).

$$
\begin{aligned}
(3) \qquad 0 = {} & \\
& \frac{d^2{x_0}^\alpha}{d\lambda^2} + \Gamma^\alpha{}_{\beta\gamma}\bigl(x_0(\lambda)\bigr)\,\frac{d{x_0}^\beta}{d\lambda}\,\frac{d{x_0}^\gamma}{d\lambda} \\
& + \epsilon\left[\frac{d^2W^\alpha}{d\lambda^2} + 2\Gamma^\alpha{}_{\beta\gamma}\bigl(x_0(\lambda)\bigr)\,\frac{d{x_0}^\beta}{d\lambda}\,\frac{dW^\gamma}{d\lambda} + \Gamma^\alpha{}_{\beta\gamma,\delta}\bigl(x_0(\lambda)\bigr)\,\frac{d{x_0}^\beta}{d\lambda}\,\frac{d{x_0}^\gamma}{d\lambda}\,W^\delta(\lambda)\right] \\
& + O(\epsilon^2).
\end{aligned}
$$

The $\epsilon^0$-order term is 0 by assumption. The requirement that the thing in brackets vanish is the equation we need to study. (Another way to get that equation is to differentiate (1) with respect to $\epsilon$.)

If we were really doing just classical applied math, we would set the expression in brackets equal to zero, draw a box around it, and relax. We want, however, to express that coordinate-dependent equation in geometrical terms.

Recall that the covariant (absolute) derivative along the curve is

$$\frac{DV^\alpha}{d\lambda} \equiv \frac{dV^\alpha}{d\lambda} + \Gamma^\alpha{}_{\beta\gamma} V^\beta U^\gamma \qquad \text{where } U^\gamma \equiv \frac{d{x_0}^\gamma}{d\lambda}\,.$$

Solve this for the coordinate derivative:

$$\frac{dV^\alpha}{d\lambda} = \frac{DV^\alpha}{d\lambda} - \Gamma^\alpha{}_{\beta\gamma} V^\beta U^\gamma. \tag{4}$$

(We will use this formula several times, in the process of eliminating coordinate derivatives in favor of covariant ones.) Let $\vec{V} = \vec{W}$ in (4) and differentiate:

$$\frac{d^2 W^\alpha}{d\lambda^2} = \frac{d}{d\lambda}\,\frac{DW^\alpha}{d\lambda} - \Gamma^\alpha{}_{\beta\gamma,\delta} W^\beta U^\gamma U^\delta - \Gamma^\alpha{}_{\beta\gamma}\,\frac{dW^\beta}{d\lambda}\,U^\gamma - \Gamma^\alpha{}_{\beta\gamma} W^\beta\,\frac{dU^\gamma}{d\lambda}\,.$$

Now apply (4) again, to $\vec{U}$, $\vec{W}$, and $\frac{D\vec{W}}{d\lambda}$ :

$$\frac{d^2W^\alpha}{d\lambda^2} = \frac{D^2W^\alpha}{d\lambda^2} - \Gamma^\alpha{}_{\beta\gamma}\,\frac{DW^\beta}{d\lambda}\,U^\gamma - \Gamma^\alpha{}_{\beta\gamma,\delta}W^\beta U^\gamma U^\delta$$
$$- \Gamma^\alpha{}_{\beta\gamma}\,\frac{DW^\beta}{d\lambda}\,U^\gamma + \Gamma^\alpha{}_{\beta\gamma}\Gamma^\beta{}_{\delta\zeta}W^\delta U^\zeta U^\gamma - \Gamma^\alpha{}_{\beta\gamma}W^\beta\,\frac{DU^\gamma}{d\lambda} + \Gamma^\alpha{}_{\beta\gamma}W^\beta\Gamma^\gamma{}_{\delta\zeta}U^\delta U^\zeta\,.$$

But the geodesic equation (1) is equivalent to

$$(1') \qquad \frac{DU^\alpha}{d\lambda} = 0.$$

Thus one of the terms we've just calculated is zero. Two of the others combine, and we get

$$(5) \qquad \begin{aligned} \frac{d^2W^\alpha}{d\lambda^2} &= \frac{D^2W^\alpha}{d\lambda^2} - 2\Gamma^\alpha{}_{\beta\gamma}\,\frac{DW^\beta}{d\lambda}\,U^\gamma \\ &\quad - \Gamma^\alpha{}_{\beta\gamma,\delta}W^\beta U^\gamma U^\delta + \Gamma^\alpha{}_{\beta\gamma}\Gamma^\gamma{}_{\delta\zeta}(W^\delta U^\zeta U^\beta + W^\beta U^\delta U^\zeta). \end{aligned}$$

Now recall that our goal is to covariantize the equation

$$0 = \frac{d^2 W^\alpha}{d\lambda^2} + 2\Gamma^\alpha{}_{\beta\gamma} U^\beta \frac{dW^\gamma}{d\lambda} + \Gamma^\alpha{}_{\beta\gamma,\delta} U^\beta U^\gamma W^\delta. \tag{3'}$$

Substituting (5) and (4) (with $\vec{V} = \vec{W}$) into (3′) and cancelling a few terms and renaming a few indices, we get finally

$$0 = \frac{D^2 W^\alpha}{d\lambda^2} + U^\mu U^\nu W^\beta \left[\Gamma^\alpha{}_{\mu\nu,\beta} - \Gamma^\alpha{}_{\beta\mu,\nu} - \Gamma^\alpha{}_{\nu\gamma}\Gamma^\gamma{}_{\beta\mu} + \Gamma^\alpha{}_{\beta\gamma}\Gamma^\gamma{}_{\mu\nu}\right]. \tag{6}$$

And our faith is rewarded! For the object in brackets in (6) is none other than $-R^\alpha{}_{\mu\nu\beta}$. Thus (6) is exactly the equation of geodesic deviation in the form (6.87).

*Modern differential-geometry approach*

**Reference:** W. L. Burke, *Applied Differential Geometry*, Sec. 61.

Recall that $\dfrac{D^2\vec{W}}{d\lambda^2} = \nabla_{\vec{U}}\nabla_{\vec{U}}\vec{W}$.

**Lemma:** $$\nabla_{\vec{U}}\vec{W} = \nabla_{\vec{W}}\vec{U}.$$

Therefore, $[\vec{V},\vec{W}] = 0$ (cf. Exercise 6.39(a), the scalar commutator rule).

**Remark:** This lemma is equivalent (via something called Frobenius's theorem) to the fact that the geodesics trace out a "ribbon" in space on which $\lambda$ and $\epsilon$ are coordinates. Our basic tacit assumption is that $\vec{W} \equiv \frac{dx}{d\epsilon}$ *exists*. This follows from the smooth dependence of solutions of ODEs on their initial data.

Proof of lemma:

$$\nabla_{\vec{U}} W^\mu = \frac{\partial}{\partial\lambda}(W^\mu) + \Gamma^\mu{}_{\alpha\beta} W^\alpha U^\beta$$
$$= \frac{\partial^2 x^\mu}{\partial\lambda\,\partial\epsilon} + \Gamma^\mu{}_{\alpha\beta} W^\alpha U^\beta,$$

which is obviously symmetrical in $\vec{U}$ and $\vec{W}$.

Proof of theorem: By the first half of the lemma,

$$\nabla_{\vec{U}}\nabla_{\vec{U}}\vec{W} = \nabla_{\vec{U}}\nabla_{\vec{W}}\vec{U}$$
$$= \nabla_{\vec{W}}\nabla_{\vec{U}}\vec{U} + \nabla_{[\vec{U},\vec{W}]}\vec{U} + R(\vec{U},\vec{U},\vec{W})$$

(by $(**)$, the vector commutator rule). But in this formula, the first term is 0 by the geodesic equation $(1')$, and the second term is 0 by the second half of the lemma. This leaves us with the geodesic equation.

If this proof went by too fast, a middle ground between it and the noncovariant proof is to write out this last calculation in components ($\nabla_{\vec{U}} = U^\alpha \nabla_\alpha$, etc.) and use the form ($*$) of the vector commutation rule.

## Normal coordinates as geodesic polar coordinates

Schutz introduced normal coordinates in a physical way, as a "local Lorentz frame": At the point $x_0$ (with coordinates 0), require

$$g_{\mu\nu}(0) = \eta_{\mu\nu} \qquad \text{and} \qquad g_{\mu\nu,\alpha}(0) = 0 \quad (\text{or } \Gamma^\alpha{}_{\beta\gamma}(0) = 0).$$

Thus

$$ds^2 = -dt^2 + \sum_j {dx_j}^2 + O\big((\text{coords})^2\big).$$

In particular, all cross terms are of second order.

Now that we understand geodesics, we can make a more geometrical, alternative construction. Corresponding to each unit tangent vector $\vec{V}$ at $x_0$ ($g_{\mu\nu}V^\mu V^\nu = \pm 1$), there is a unique geodesic that starts from 0 in the direction $\vec{V}$ — i.e., has the initial data

$$x^\mu(0) = 0, \qquad \frac{dx^\mu}{ds}(0) = V^\mu.$$

Each point near $x_0$ on such a curve can be labeled by (1) its arc length, $s \equiv r$, from the origin, and (2) the vector $\vec{V}$, which is essentially $n-1$ angular coordinates ($n =$ dimension). In the Riemannian case we have this sort of picture:

Eventually the geodesics may cross each other — this phenomenon is called a *caustic* — but near $x_0$ they neatly cover a neighborhood. In the Lorentzian case the spheres become hyperboloids, and the coordinates are singular on the light cone.

However, interpreting $r$ as a radial coordinate and adopting suitable angular coordinates for $\vec{V}$, we recover a nonsingular Cartesian coordinate system by a hyperbolic analogue of the polar-to-Cartesian transformation. In two-dimensional space-time, this transformation is

$$t = r \sinh \chi, \qquad x = r \cosh \chi$$

in one of the quadrants, and similar formulas in the others. (These formulas should be familiar from previous discussions of rapidity and of uniform acceleration.)

Let us look at the form of the metric in the polar coordinates. I will concentrate on the Riemannian case for simplicity. The coordinates are $r$, $\theta_1$, ..., $\theta_{n-1}$, where the $\theta$'s are angular coordinates for the unit vector $\vec{V}$ regarded as a point on the unit sphere. The space of tangent vectors at $x_0$ can be thought of as a copy of *Euclidean* $\mathbf{R}^n$. In the Lorentzian case the story is the same, except that one of the angles is a hyperbolic rapidity variable $\chi$ and the tangent space is Minkowskian $\mathbf{R}^n$.

We now ask: What is $ds^2$ in these coordinates? Well, for displacements along the radial geodesics, the arc length just equals the coordinate increment. Thus

$$ds^2 = dr^2 + (\text{terms involving } d\theta_j).$$

I claim:

**Theorem:**

(1) The geodesics are *orthogonal* to the surfaces of constant $r$ (the spheres or hyperboloids). Therefore, the $dr\,d\theta_j$ terms are *identically zero*.

(2) The purely angular part of the metric coincides with the angular part of the metric of Euclidean space, up to corrections of second order in $r$ (if the metric is sufficiently smooth).

That is,

$$ds^2 = dr^2 + r^2[d\Omega^2 + O(r^2)],$$

where only angular differentials appear in the error term (and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ in the usual polar coordinates in $\mathbf{R}^3$, for example).

In the Lorentz case, the corresponding statement is

$$ds^2 = \pm dr^2 + r^2[d\Omega_\pm{}^2 + O(r^2)],$$

where $d\Omega_\pm$ is an appropriate hyperboloidal metric and the sign depends on whether the geodesic to the point in question is spacelike or timelike.

Here is a *crude*, intuitive proof of (1) in the Riemannian case: Suppose the geodesic at $\mathcal{P}$ does *not* meet the sphere normally, Draw a curve that joins the geodesic at a point $\mathcal{R}$ very near $\mathcal{P}$ and *does* meet the sphere normally, at a point $\mathcal{Q}$. We may assume that the region containing $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$ is so small that the geometry is essentially Euclidean there, and that $\mathcal{QR}$ and $\mathcal{QP}$ are approximately straight lines. Then $\mathcal{PQR}$ is approximately a right triangle, and by the Pythagorean theorem $\mathcal{RQ}$ is shorter than $\mathcal{RP}$. Hence $0\mathcal{RQ}$ is shorter than $r$, the geodesic radius of the sphere (the length of both $0\mathcal{P}$ and $0\mathcal{Q}$. But this contradicts the fact that $0\mathcal{Q}$ is the shortest path from 0 to $\mathcal{Q}$.

A more convincing and general proof also makes use of the variational characterization of geodesics. Recall that the affinely parametrized geodesic $0\mathcal{P}$ stationarizes the action

$$\sigma \equiv \int_0^1 \frac{1}{2}\, g_{\mu\nu}\bigl(x(\lambda)\bigr)\, \frac{dx^\mu}{d\lambda}\, \frac{dx^\nu}{d\lambda}\, d\lambda,$$

and that on the geodesic, $\sigma = \frac{1}{2} r^2$. (The normalization condition that $\lambda$ runs from 0 to 1 is equivalent to the condition that the length of the initial tangent vector $\frac{dx^\mu}{d\lambda}(0)$ is equal to $r$, the length of the geodesic segment in question. This is a change from the convention used earlier, where the tangent vector was assumed to be a unit vector.)

Consider the variation of $0\mathcal{P}$ to $0\mathcal{Q}$, $\mathcal{Q}$ being a point very near $\mathcal{P}$ on the same geodesic sphere. Thus $\delta\sigma = 0$. To calculate the variation from the integral, note that for fixed $\lambda$, $x(\lambda)$ is determined by $\mathcal{Q}$, since the geodesic $0\mathcal{Q}$ is unique as long as we stay inside a small enough neighborhood of 0. So we get

$$\delta\sigma = \int \left[ \frac{1}{2} \frac{\partial g_{\mu}\nu}{\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial Q^{\tau}} \delta Q^{\tau} \frac{dx^{\mu}}{d\lambda} \frac{dx^{\nu}}{d\lambda} + g_{\mu\nu} \frac{dx^{\mu}}{d\lambda} \frac{\partial}{\partial Q^{\tau}} \left( \frac{dx^{\nu}}{d\lambda} \right) \delta Q^{\tau} \right],$$

where $\{Q^{\tau}\}$ are coordinates of $\mathcal{Q}$ (in any system). Since $\lambda$ and $\mathcal{Q}$ are independent variables (on which $x$ depends),

$$\frac{\partial}{\partial Q^{\tau}} \left( \frac{dx^{\nu}}{d\lambda} \right) = \frac{d}{d\lambda} \left( \frac{\partial x^{\nu}}{\partial Q^{\tau}} \right).$$

This allows us to integrate by parts in the second term of $\delta\sigma$, thereby revealing (after renaming $\nu$ as $\rho$) a common factor $\frac{\partial x^\rho}{\partial Q^\tau}\,\delta Q^\tau$ in the first and second terms. Moreover, the total expression multiplying this object *vanishes*, because it is just the geodesic equation. (Recall that the derivation of the geodesic equation from $\sigma$ involves essentially this same calculation, except that there the variation vanished at *both* endpoints and was not itself a geodesic.) So we are left with the endpoint terms:

$$0 = \delta\sigma = g_{\mu\nu}\,\frac{dx^\mu}{d\lambda}\,\frac{\partial x^\nu}{\partial Q^\tau}\,\delta Q^\tau\bigg|_0^{\mathcal{P}}\,.$$

The contribution from the endpoint 0 vanishes since that end of the geodesic is fixed. At the other end, $x$ *is* $\mathcal{Q}$, so $\frac{\partial x^\nu}{\partial Q^\tau} = \delta^\nu_\tau$ and

$$0 = g_{\mu\nu}\,\frac{dx^\mu}{d\lambda}\,\delta Q^\nu.$$

This conclusion has been proved for all $\delta Q^\nu$ parallel to the sphere. Therefore, the tangent vector to the geodesic is perpendicular to the sphere, QED.

**Remark.** A generalization of this argument proves that $g_{\mu\nu}\frac{dx^\nu}{d\lambda} = \nabla_\mu\sigma$. In words: The gradient of the action $\sigma(x, x_0)$ with respect to $x$ is equal (up to index raising) to the vector tangent to the geodesic segment from $x_0$ to $x$ and with length equal to the length of the segment. (This vector is in the tangent space at $x$, and it points *away* from $x_0$.) This frequently used formula is stated without proof in my book (p. 177), after the observation that it is obvious in flat space. I thank Frank Molzahn for help on this point.

We still need to prove property (2) of the theorem. My proof of this is suggested by, but different from, the appendix of R. N. Pederson, *Commun. Pure Appl. Math.* **11**, 67 (1958). (He proves a weaker conclusion from a weaker smoothness hypothesis.)

Write the solution of the geodesic equation as

$$x^\mu = rV^\mu + O(r^2). \tag{$*$}$$

(That is what it means for $\vec{V}$ to be the tangent vector to the geodesic, with parameter $r$.) Regard this as the transformation from polar normal coordinates into the original, arbitrary coordinates. Since

$$dx^\mu = \frac{\partial x^\mu}{\partial r}\,dr + \sum_j \frac{\partial x^\mu}{\partial \theta^j}\,d\theta^j\,,$$

we have

$$ds^2 = dr^2 + 0\,dr\,d\theta^j + \sum_{j,k} g_{\mu\nu}\,\frac{\partial x^\mu}{\partial \theta^j}\,\frac{\partial x^\mu}{\partial \theta^k}\,d\theta^j\,d\theta^k$$

(where the form of the $dr$-dependent terms follows from previous considerations). But according to $(*)$,

$$\frac{\partial x^\mu}{\partial \theta^j} = r\,\frac{\partial V^\mu}{\partial \theta^j} + O(r^2).$$

When we substitute this into the formula for $ds^2$, the first term yields $r^2\,d\Omega^2$. The error term yields something of order $O(r^4) = r^2 O(r^2)$, which is what we want to see. [Look back at the statement of property (2).] Superficially the cross term is $O(r^3)$. However, we shall show that to lowest nontrivial order in $r$, the $O(r^2)$ term in $(*)$ is orthogonal to the other term. Thus the cross term vanishes to $r^3$ order; it is actually $O(r^4)$ as needed.

To prove this orthogonality assertion, write the Taylor expansion of the geodesic as

$$x^\mu(r) = rV^\mu(0) + \frac{1}{2}r^2U^\mu + O(r^3),$$

where $\vec{U}$ is at present unknown to us, but we want to prove it orthogonal to $\vec{V}$. We have

$$V^\mu(r) \equiv \frac{dx^\mu}{dr} = V^\mu(0) + rU^\mu + O(r^2),$$

hence

$$\mathbf{g}\big(\vec{V}(r), \vec{V}(r)\big) = 1 + 2rV^\mu U_\mu + O(r^2).$$

But recall that for an affinely parametrized geodesic, the length of the tangent vector is always *exactly* equal to 1. Therefore the term $2rV^\mu U_\mu$ must vanish, QED. (This is essentially the same argument that shows that 4-acceleration in special relativity is orthogonal to 4-velocity.)

In summary, we have shown that

$$ds^2 = (\text{flat metric in polar coords.}) + O(r^4).$$

When we convert from polar to Cartesian coordinates we lose two powers of $r$ from the erstwhile angular terms:

$$ds^2 = (\text{flat metric in Cartesian coords.}) + O(r^2).$$

Thus the new Cartesian coordinates are [the Riemannian analogue of] *local inertial coordinates* as defined by Schutz.

# Physics in Curved Space *(Chapters 7 and 8)*

## The Strong Equivalence Principle[2]

By "strong equivalence principle" I mean the corollary Schutz draws from the Einstein equivalence principle IV$'$. It is the assertion that the only gravitational interaction is that obtained from special-relativistic equations of motion (of matter, fields, etc.) by replacing $\eta^{\alpha\beta}$ by $g^{\alpha\beta}$ and $\partial_\mu$ by $\nabla_\mu$. It is the counterpart of *minimal coupling* of electromagnetism: The only electromagnetic interaction of matter is that obtained by replacing $\partial_\mu$ by $\partial_\mu + ieA_\mu$ in the Schrödinger equation, Klein–Gordon field equation, etc.

[2]This section of the book has changed a lot in the new edition, so these notes may soon change, too.

In my opinion, the strong equivalence principle is not a dogma, but a strategy: a tentative assumption made to reduce the number of arbitrary parameters in the theory. This is an instance of *Occam's razor*: Don't complicate your theory unless and until experiment forces you to do so.

For example, consider the massless Klein–Gordon equation satisfied by a scalar field:

$$\Box\phi \equiv \eta^{\alpha\beta}\partial_\alpha\partial_\beta\phi = 0.$$

The minimal generalization to curved space is

$$\Box_g\phi \equiv g^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi = 0.$$

However, what is wrong with

$$g^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi + \xi R\phi = 0,$$

where $\xi$ is some numerical constant? (The curvature term vanishes in flat space.) In fact, there are theoretical reasons (conformal invariance) for preferring the value $\xi = \frac{1}{6}$ to $\xi = 0$!

Minimal coupling also contains an ambiguity: Should $\partial_\alpha \partial_\beta$ be replaced by $\nabla_\alpha \nabla_\beta$ or by $\nabla_\beta \nabla_\alpha$? Their difference involves $R^\mu{}_{\nu\alpha\beta}$, in general. (In the example, this didn't appear, for two reasons.)

This ambiguity has an analogue in electromagnetism. Consider the Schrödinger equation for a nonrelativistic particle with spin $\frac{1}{2}$. The wave function is a two-component spinor. Such objects are acted upon by the famous *Pauli matrices*, satisfying

$$\sigma_1 \sigma_2 = i\sigma_3 = -\sigma_2 \sigma_1 \,, \text{etc.}, \qquad {\sigma_j}^2 = 1.$$

It follows that in the kinetic-energy term of the classical Hamiltonian,

$$(\mathbf{p} \cdot \sigma)^2 = \mathbf{p}^2. \qquad (*)$$

(A $2 \times 2$ identity matrix is implicit on the right side of this equation and in similar expressions hereafter.) In quantum mechanics, $p_j$ gets replaced by $-i\partial_j$ . So far there is no problem. However, when we add a magnetic field, the left side of $(*)$ becomes

$$[(\mathbf{p} - ie\mathbf{A}) \cdot \sigma]^2 = \left[\sum_{j=1}^{3}(-i\partial_j + eA_j)\sigma_j\right]^2$$

with typical term

$$(-i\partial_3 + eA_3)^2 + \sigma_1\sigma_2\big[(-i\partial_1 + eA_1)(-i\partial_2 + eA_2) - (-i\partial_2 + eA_2)(-i\partial_1 + eA_1)\big].$$

The expression in brackets boils down to

$$-ie(\partial_1 A_2 - \partial_2 A_1) = -ieB_3 \, .$$

Therefore,

$$[(\mathbf{p} - ie\mathbf{A}) \cdot \vec{\sigma}]^2 = (\mathbf{p} - ie\mathbf{A})^2 + e\mathbf{B} \cdot \vec{\sigma}.$$

The Hamiltonian with the field turned on seems to depend upon an arbitrary decision about how to write down the Hamiltonian for the case with the field turned off! In fact, the version including the magnetic moment term, $e\mathbf{B}\cdot\vec{\sigma}$, is the *correct* one, as determined by experiment or by reduction from the Dirac equation. (Note that in the modern understanding of gauge theories, $eF_{\mu\nu}$ is the "curvature" associated with the "electromagnetic covariant derivative" $\vec{\partial} + ie\mathbf{A}$. Thus the relationship of this example to the ambiguity in the strong equivalence principle is rather close.)

So far I have been talking about *fundamental* dynamical equations. Schutz's examples of the strong equivalence principle are all *secondary* equations, where (in my opinion) the SEP is more a *definition* than a physical assumption.

1. **Conservation of particles.**

$$(nU^\alpha)_{;\alpha} = 0.$$

Suppose we generalized this to $(nU^\alpha)_{;\alpha} = \lambda R$. Only $\lambda = 0$ would describe a theory in which the number of particles is conserved. If we start with conservation, then the equation with $\lambda = 0$ follows from Gauss's theorem, (6.45).

2. **Conservation of energy.**

$$\nabla_\mu T^{\mu\nu} = 0.$$

It can be shown (as a form of *Noether's theorem*) that this must be true in any generally covariant theory derived from a (covariant) action principle. In practice, the physical question is not whether the stress tensor is conserved, but rather what the conserved stress tensor *is* for a given theory. For the fluid example, the conservation of (7.7) follows from the properties of the quantities in it; it is not a separate postulate.

The Newtonian gravitational potential equations are given as [(8.1–2)]

$$\nabla^2\phi = 4\pi G\rho; \qquad \phi = -\frac{Gm}{r} \quad \text{for a point mass.}$$

This historical definition of $G$ is clearly off from the "natural" or "fundamental" definition by a factor of $4\pi$; logically the $4\pi$ belongs not in the fundamental differential equation, but in the *solution* of a problem with *spherical symmetry*:

$$\nabla^2\phi = G\rho; \qquad \phi = -\frac{Gm}{4\pi r} \quad \text{for a point mass.}$$

The same unfortunate tradition survives in traditional CGS electromagnetic units; it has been cured in Lorentz–Heaviside units (and in SI units, but those have a worse ingredient, the gratuitous constants $\epsilon_0$ and $\mu_0$).

In passing to general relativity, some additional factors of 2 accumulate. Thus Einstein's basic gravitational field equation is [(8.10)]

$$G_{\alpha\beta} = 8\pi G T_{\alpha\beta} \,,$$

and its linearized (weak-field) version in Lorentz gauge is [(8.42)]

$$\Box \overline{h}^{\mu\nu} = -16\pi G T^{\mu\nu}.$$

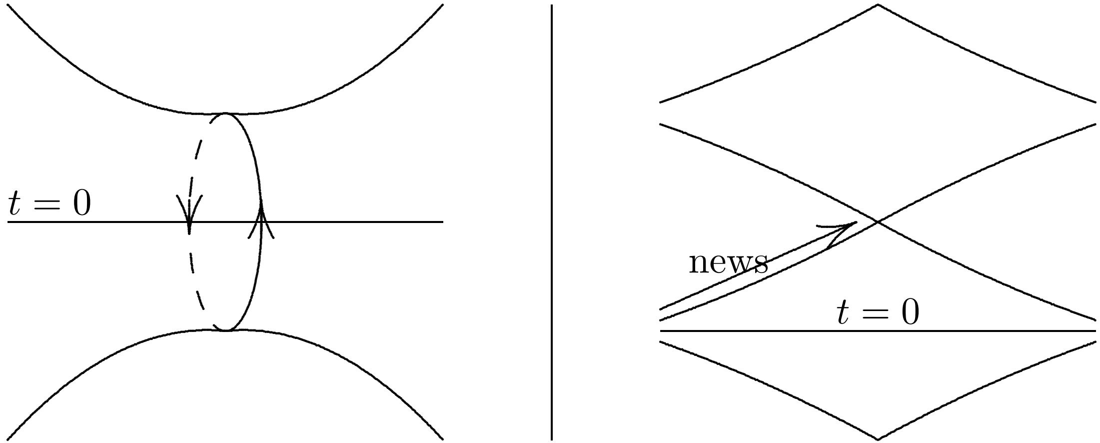
In these equations I have reinserted a constant $G$, where Schutz chose units so that $G = 1$. One can also choose $G = 1/4\pi$, $1/8\pi$, or $1/16\pi$, according to taste. Be prepared to encounter many conventions in later life.

Incidentally, a wave equation such as (8.42) has solutions with the usual structure for the solution space of a linear equation:

$$\text{any particular solution} + \text{solution of homogeneous equation.}$$

Moreover, for the wave equation in particular (or any other second-order hyperbolic equation) the physically preferred solution is usually the one in which the source ($T$) affects the field ($\overline{h}$) only in the future. (Think of $\overline{h}$ as analogous to electromagnetic field and $T$ as analogous to electric charge and current density. Radiation from a moving charge will show up only inside and on the light cones of the space-time points where the moving charge is.) When a problem has been properly posed (e.g., the gauge freedom discussed below has been taken care of), for a given source (satisfying mild technical conditions) there will exist a unique solution with this property, the *retarded solution*. However, the most general solution also contains a homogeneous piece, which does not vanish in the far past. This term represents *incident waves*, radiation that was already present in the system, not created by the source. In principle, you can have gravitational radiation without having any matter source in the region of space considered.

In general, the solution of a wave equation (even a nonlinear one, such as the full Einstein equation) is uniquely determined by giving the value of the field and its time derivative at one time. In curved space, this means the field and its normal derivative on a spacelike hypersurface. (Here I am sweeping a whole subject under the rug. The statement I have just made is true (by definition) only if the hypersurface is a *Cauchy hypersurface* — big enough so that data on it determine the solution uniquely, but not so big that data on it can be inconsistent (preventing existence). Whether such a hypersurface exists is a nontrivial question about the global geometry of the space-time. For example, two-dimensional DeSitter space turned on its side has an existence problem if the periodic time coordinate is taken seriously, and a uniqueness problem if it is "unwrapped".)

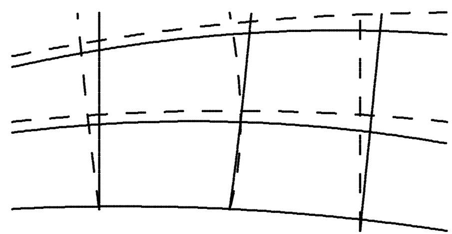



## Counting degrees of freedom

How many independent components does the gravitational field have? This is a very subtle question. (Let's consider only space-time dimension 4.)

At the most naive level, the metric $\{g_{\mu\nu}\}$ is a $4 \times 4$ matrix, so its components comprise 16 fields. However, this matrix is symmetric, so it's immediately obvious that there are only 10 independent components.

But this is not the end of the story. Consider initial data on a certain hypersurface, and contemplate your right to change the coordinate system *off* the hypersurface.



From the point of view of the coordinate grid in which you do your calculations, the solution will look different in the two systems. Therefore, contrary to appearance, the 10 hyperbolic equations plus initial data must *not* uniquely determine the 10 components of the solution. This means that in some sense the

10 equations are not independent. The resolution of this conundrum is that the energy-momentum conservation law, $T^{\mu\nu}{}_{;\nu} = 0$, is 4 linear combinations of the derivatives of the 10 equations. (On the field side of the equations, this constraint is the *contracted Bianchi identity*, $G^{\mu\nu}{}_{;\nu} = 0$.) This corresponds neatly to the 4 freely choosable functions in a coordinate transformation. Therefore, we expect that only 6 components of the metric have true physical meaning, and that the Einstein equations need to be supplemented by 4 conditions restricting the choice of coordinates before the full set of equations will determine the metric tensor uniquely from its initial data. The 4 coordinate conditions are analogous to gauge conditions in electromagnetism.

However, this is still not all. There is a sense in which the number of independent dynamical degrees of freedom of the gravitational field is only *two*, not six. To clarify both this and the previous reduction from 10 to 6, I will first consider the electromagnetic analogy in more detail.

Recall from Part 5 of our investigation of "Special Relativity and Electromagnetism" that the equation of motion of the vector potential $A^\alpha = (-\phi, \mathbf{A})$ is

$$-\partial^\mu \partial_\mu A^\alpha + \partial^\alpha \partial_\mu A^\mu = J^\alpha .$$

(I use Lorentz–Heaviside units so I can ignore the factor $4\pi$.) The spatial ($\alpha = j$) component of this equation is

$$-\partial^\mu \partial_\mu \mathbf{A} + \nabla(-\partial_t \phi + \nabla \cdot \mathbf{A}) = \mathbf{J}. \tag{1}$$

The time component is

$$\partial^\mu \partial_\mu \phi - \partial_t(-\partial_t \phi + \nabla \cdot \mathbf{A}) = \rho. \tag{2}$$

The left side of (2) can be written

$$\nabla^2 \phi - \partial_t(\nabla \cdot \mathbf{A}) = \nabla \cdot (\nabla \phi - \partial_t \mathbf{A}) = \nabla \cdot \mathbf{E},$$

so (2) is just "Gauss's Law".

From Part 8 of the electromagnetism paper, the current conservation law, $\partial_t \rho + \nabla \cdot \mathbf{J} = 0$, follows automatically from the structure of the Maxwell equations. This was easy to see when the equations were written in terms of $F_{\alpha\beta}$; it can be verified fairly quickly from (1) and (2). Thus only three of the four equations are independent. This reflects the freedom to make a gauge transformation, $A'_\alpha = A_\alpha + \partial_\alpha \chi$; inherently, one of the components of $A$ is arbitrary. Indeed, charge conservation and gauge invariance are closely related. One can show that the conservation law follows from the gauge invariance of the electromagnetic action integral introduced in Part 16.

**Remark:** If we perform a Fourier transformation in all four space-time coordinates, partial differentiation goes over into multiplication by a component of the Fourier variable (the wave vector $k_\alpha = (\omega, \mathbf{k})$, identified quantum-mechanically with a four-momentum vector). From this point of view the conservation law is

$k_\alpha \hat{J}^\alpha(k) = 0$, which says that at each point in $k$-space, the [Fourier transform of the] current vector is perpendicular to $k$ (with respect to the Lorentz inner product). In the Fourier picture, therefore, the 4 differential equations are literally linearly dependent: this particular linear combination of them vanishes. Only the component of Maxwell's equation perpendicular to the wave vector is nontrivial. (It may seem that there is a swindle here. When we differentiate a differential equation with respect to one of its dependent variables, we do not get a precisely equivalent equation: some information about initial data is thrown away. However, in Fourier space we are just multiplying by a variable, and we could later divide by it and recover the original equation. The resolution of this paradox is that the Fourier treatment tacitly assumes that the functions in the problem are decaying at infinity, so that their Fourier transforms are well defined. This is a restriction on the solutions of the equation that we may not be able to uphold in general. Such technicalities are beyond the scope of this discussion, and consequently a lot of my remarks will have the ring of "numerology", not rigorous mathematical statements.)

So, gauge freedom cuts the number of independent equations of motion down from 4 to 3, and likewise the number of physically significant components of the electromagnetic [potential] field. However — and here comes the main point — let's look more closely at (2). It does not involve any second-order time derivatives; in terms of **E**, it does not involve any time derivatives at all. Therefore, it is not really an equation of motion at all, but a *constraint on the allowed initial data* for the problem. Not all possible initial data (fields and their first time derivatives) are allowed in this theory. In principle, (2) can be solved for one field in terms of the others. This is *in addition* to the one field that can be freely specified by a choice of gauge. Therefore, of the four fields, there are really only *two* that are independent and truly physical. From a quantum-theoretical point of view, these correspond to the two polarization states of a photon. After giving the energy and direction of motion (hence the momentum) of a photon, its state is completely specified by giving its polarization, and there are only two choices, not three or four.

To investigate this in more detail, let's impose Coulomb gauge,

$$\nabla \cdot \mathbf{A} = 0,$$

where the effect is especially striking. In Fourier space this condition is $\mathbf{k} \cdot \hat{\mathbf{A}}(k) = 0$. That is, in this gauge the *longitudinal* part of $\hat{\mathbf{A}}$ is not only not an independent dynamical object, it is nonexistent. $\hat{\mathbf{A}}(k)$ has only two components, its perpendicular or *transverse* part. Since we are in 3 dimensions, the latter can be extracted by taking the vector cross product $\mathbf{k} \times \hat{\mathbf{A}}$, which translates back to $\mathbf{x}$-space as $\nabla \times \mathbf{A} \equiv \mathbf{B}$.

In Coulomb gauge, (2) simplifies to

$$\nabla^2 \phi = \rho,$$

which contains no time derivatives at all! If we assume $\phi$ vanishes at infinity, this can be solved immediately by the Coulomb potential:

$$\phi(\mathbf{x}) = -\frac{1}{4\pi} \int \frac{\rho(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} \, d^3y.$$

(More generally, we could add to $\phi$ any time-dependent solution of the three-dimensional Laplace equation. Such a term is the time derivative of another function of the same type, $\chi$, and then $\mathbf{A}' = \mathbf{A} + \nabla\chi$ still satisfies $\nabla\cdot\mathbf{A}' = 0$. Thus such a term is just another part of the gauge freedom and can be disregarded.)

We have now determined both $\nabla\cdot\mathbf{A}$ and $\phi$ without solving a single hyperbolic equation. For consistency, the other Maxwell equation, (1), must provide exactly the information needed to solve for the remaining two components of the field, the transverse part of $\mathbf{A}$. The Coulomb-gauge form of (1) is

$$-\partial^\mu\partial_\mu\mathbf{A} - \nabla(\partial_t\phi) = \mathbf{J}.$$

Although this looks like three equations, it is really only two, since the divergence of it contains no new information:

$$-\partial^\mu\partial_\mu\nabla\cdot\mathbf{A} - \nabla^2(\partial_t\phi) = \nabla\cdot\mathbf{J};$$

$$-\partial_t(\nabla^2\phi) = -\partial_t\rho;$$

this is just the derivative of (2). We can exhibit the two genuine equations by taking the cross product with $\mathbf{k}$ in the Fourier representation. The result is equivalent to

$$\partial^\mu\partial_\mu \mathbf{B} = -\nabla\times\mathbf{J}.$$

(The reason why $\mathbf{B}$ has only two independent components is that $\nabla\cdot\mathbf{B} = 0$.)

This whole analysis could be repeated in *temporal gauge*, defined by the condition $\phi = 0$; the results are quite similar. However, the case of Lorentz gauge is harder to analyze, because the constraint equation (2) is disguised as a hyperbolic equation, $\Box\phi = \rho$ (see Part 7).

Gravity is harder still, because (1) the equations are nonlinear; (2) there are 10 fields, not 4, forming a tensor, not a vector; (3) there are 4 conservation laws and 4 gauge choices, not 1. However, the results are analogous. As previously mentioned, the gauge freedom and associated conservation laws cut down the degrees of freedom from 10 to 6. In addition, there are 4 initial-data constraints analogous to (2); they can be identified with the Einstein equations with left-hand sides $G_{00}$ and $G_{0j}$ , for these contain no second-order time derivatives of the metric tensor components (cf. Exercise 8.9). As a result the dynamical degrees of freedom are cut down from 6 to 2, corresponding to two polarization states of the graviton.

## More abou the number of degrees of freedom of a gauge theory

Let us work in the Fourier picture (see Remark above). In a general gauge, the Maxwell equation for the vector potential is

$$-\partial^\mu \partial_\mu A^\alpha + \partial^\alpha \partial_\mu A^\mu = J^\alpha. \tag{3}$$

Upon taking Fourier transforms, this becomes

$$k^\mu k_\mu A^\alpha - k^\alpha k_\mu A^\mu = J^\alpha, \tag{3'}$$

where $\vec{A}$ and $\vec{J}$ are now functions of the 4-vector $\vec{k}$. (One would normally denote the transforms by a caret ($\hat{A}^\alpha$, etc.), but for convenience I won't.) The field strength tensor is

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha, \tag{4}$$

or

$$F^{\alpha\beta} = ik^\alpha A^\beta - ik^\beta A^\alpha. \tag{4'}$$

The relation between field and current is (factor $4\pi$ suppressed)

$$J^\alpha = \partial_\beta F^{\alpha\beta}, \tag{5}$$

or

$$J^\alpha = ik_\mu F^{\alpha\mu}. \tag{5'}$$

Of course, (4′) and (5′) imply (3′).

(3′) can be written in matrix form as

$$\vec{J} = M\vec{A}, \tag{6}$$

$$M(\vec{k}) = k^\mu k_\mu\, I - \vec{k} \otimes \tilde{k} = \begin{pmatrix} \vec{k}^2 - k^0 k_0 & -k^0 k_1 & -k^0 k_2 & -k^0 k_3 \\ -k^1 k_0 & \vec{k}^2 - k^1 k_1 & -k^1 k_2 & -k^1 k_3 \\ -k^2 k_0 & -k^2 k_1 & \vec{k}^2 - k^2 k_2 & -k^2 k_3 \\ -k^3 k_0 & -k^3 k_1 & -k^3 k_2 & \vec{k}^2 - k^3 k_3 \end{pmatrix} \tag{7}$$

Consider a generic $\vec{k}$ (not a null vector). Suppose that $\vec{A}$ is a multiple of $\vec{k}$:

$$A^a(\vec{k}) = k^\alpha \chi(\vec{k}). \tag{6′}$$

Then it is easy to see that $\vec{A}$ is in the kernel (null space) of $M(\vec{k})$; that is, it yields $\vec{J}(\vec{k}) = 0$. (In fact, by (4′) it even yields a vanishing $F$.) Conversely, every vector in the kernel is of that form, so the kernel is a one-dimensional subspace. Back in space-time, these observations correspond to the fact that a vector potential of the form

$$\vec{A} = \nabla\chi \tag{8}$$

is "pure gauge". This part of the vector potential obviously cannot be determined from $\vec{J}$ and any initial data by the field equation, since it is entirely at our whim. (Even if the Lorenz gauge condition is imposed, we can still perform a gauge transformation with $\chi$ a solution of the scalar wave equation.)

Now recall a fundamental theorem of finite-dimensional linear algebra: *For any linear function, the dimension of the kernel plus the dimension of the range equals the dimension of the domain.* In particular, if the dimension of the domain equals the dimension of the codomain (so that the linear function is represented by a square matrix), then the dimension of the kernel equals the codimension of the range (the number of vectors that must be added to a basis for the range to get a basis for the whole codomain). Thus, in our situation, there must be a one-dimensional set of vectors $\vec{J}$ that are left out of the range of $M(\vec{k})$. Taking the scalar product of $\vec{k}$ with $(3')$, we see that

$$k_\alpha J^\alpha = 0 \tag{9'}$$

is the necessary (and sufficient) condition for (6) to have a solution, $\vec{A}$. In space-time, this condition is the conservation law,

$$\partial_\alpha J^\alpha = 0. \tag{9}$$

$(9')$ can be solved to yield

$$\rho = -\frac{\mathbf{k}\cdot\mathbf{J}}{k_0}\,. \tag{10''}$$

In terms of $\vec{A}$, the right-hand side of $(10'')$ cannot contain ${k_0}^2$ (since $(3')$ is quadratic in $\vec{k}$); that is, the Fourier transform of $(10'')$ is a linear combination of components of the field equation that does not contain second-order time derivatives. In fact, a few more manipulations show that

$$\rho = i\mathbf{k} \cdot \mathbf{E}, \tag{10$'$}$$

whose transform is

$$\rho = \nabla \cdot \mathbf{E}. \tag{10}$$

That is, the conservation law is essentially equivalent (up to the "swindle" mentioned in the Remark) to the Gauss law, which is a *constraint* on the allowed initial data (including first-order time derivatives) for $\vec{A}$.

Conclusion: At each $\vec{k}$ (hence at each space-time point) there are only *two* independent physical degrees of freedom, not four or even three. One degree of freedom is lost to the gauge ambiguity; another is cut out of the space of candidate solutions by the constraint related to the conservation law. But by Noether's theorem, the conservation law is itself a consequence of the gauge invariance. In the Fourier picture the fact that degrees of freedom are lost in pairs is consequence of the dimension theorem for linear functions.

## Two analogies between electromagnetism and gravity

Solid lines indicate the gauge-transformation analogy. Dashed lines indicate the covariant-derivative analogy. Single-shafted arrows indicate differentiation. Double-shafted arrows indicate a trace operation.

$$
\begin{tikzcd}
 & & & F_{[\alpha\beta,\gamma]} = 0 \arrow[ddr, dashed, "\text{Bianchi identity}"] & \\
\theta \arrow[r] \arrow[d, dash, "\text{gauge group}"'] & A_\alpha \arrow[r] \arrow[d, dash, "\text{potential}"'] \arrow[dr, dashed, dash, "\text{connection}"] & F_{\alpha\beta} \arrow[r] \arrow[ur] \arrow[d, dash, "\text{field strength}"'] \arrow[dr, dashed, dash, "\text{curvature}"] & J^\alpha \arrow[r] \arrow[dd, dash, bend left, "\text{source}"] & J^\alpha{}_{,\alpha} = 0 \arrow[dd, dash, bend left, "\text{conservation law}"] \\
x^{\bar\alpha} \arrow[r] & g_{\alpha\beta} \arrow[r] & \Gamma^\alpha_{\beta\gamma} \arrow[r] & R^\alpha{}_{\beta\gamma\delta} \arrow[r] \arrow[d, Rightarrow] & R^\alpha{}_{\beta[\gamma\delta;\epsilon]} \arrow[d, Rightarrow] \\
 & & & T^{\alpha\beta} & T^{\alpha\beta}{}_{;\beta} = 0
\end{tikzcd}
$$

# Cosmology *(Chapter 12)*

## Basics of Robertson–Walker cosmology[3]

My goal here is to fill in some details in Sec. 12.3, especially the derivation of the crucial equation (12.46). The main steps in the argument are these:

1. Assume that (in the mean rest frame of the galaxies) the universe is *spatially* homogeneous and isotropic, but is changing in time (in an isotropic way, so that $g_{0j} = 0$).

[3] This chapter of the book has changed a lot in the new edition, so these notes will probably receive further changes, too.

2. Conclude that (12.6)

$$ds^2 = -dt^2 + A(t)^2\, h_{jk}(\mathbf{x})\, dx^j\, dx^k,$$

where $\mathbf{h}$ is a time-independent homogeneous, isotropic Riemannian metric on a three-dimensional manifold. (I write $A$ instead of $R$ to avoid confusion with the curvature.)

3. Classify the possibilities for $\mathbf{h}$. There are three main possibilities:

$k = 0$: flat $\mathbf{R}^3$

$k = -1$: 3-dimensional hyperboloid with negative curvature (topologically $\mathbf{R}^3$)

$k = 1$: 3-sphere with positive curvature

(There are other possibilities obtained by identifying points in one of these models (e.g., a torus or cylinder for $k = 0$). This would not change the dynamics for $A(t)$.)

4. Calculate the Einstein tensor. The only component we need to look at in detail is (12.50),
$$G_{tt} = 3\left[\left(\frac{A'}{A}\right)^2 + \frac{k}{A^2}\right].$$

5. Find the form of the stress tensor. (Here is the first part that requires my commentary.) By symmetry, **T** must have the perfect-fluid form (4.36) in a comoving local inertial frame:
$$T^{\alpha\beta} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}.$$

For a metric of the form (12.6), this implies that

$$T^{00} = \rho, \qquad T^{jk} = p\, g^{jk}.$$

(One way of seeing this is to take the covariant special-relativistic formula (4.37),

$$T^{\alpha\beta} = (\rho + p)U^\alpha U^\beta + p\eta^{\alpha\beta},$$

replace $\eta^{\alpha\beta}$ by $g^{\alpha\beta}$ to generalize to curved space, and set $U^0 = (1, 0, 0, 0)$ since the coordinate frame is comoving with the matter (i.e., $g_{0j} = 0$, $g_{00} = -1$).) Note that we have made no assumption on how $\rho$ and $p$ are related to each other (an *equation of state*). Therefore, the formulation so far is applicable to any kind of matter: cold dust, radiation, hot massive particles, or an exotic nonclassical fluid of some kind.

6. Examine the Einstein equation, $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$. The off-diagonal components are $0 = 0$. The three spatial diagonal components are all the same,

and they are related to the temporal diagonal component by the Bianchi identity (whose detailed investigation I am postponing). Therefore, there is only *one* independent component. It turns out that the spatial equation is essentially the time derivative of the temporal one, and hence contains *less* information, since a constant of integration is lost. The evolution of the universe is therefore determined by the equation $G_{tt} = 8\pi T_{tt}$ , which is (12.54), or

$$\left(\frac{A'}{A}\right)^2 = -\frac{k}{A^2} + \frac{8\pi}{3}\,\rho$$

— together with an equation of state to fix the behavior of $\rho(t)$.

7. At this point the Bianchi identity (or local energy conservation law) becomes useful. Schutz states it as (12.21)

$$\frac{d}{dt}(\rho A^3) = -p\,\frac{d}{dt}(A^3),$$

and pronounces it "easy to show". Let us show it. The conservation law is $T^{\mu\nu}{}_{;\nu} = 0$. Only the time component ($\mu = 0$) is nontrivial in this spatially homogeneous model. That component is

$$\begin{aligned}
0 &= T^{t\nu}{}_{;\nu} \\
&= T^{t\nu}{}_{,\nu} + T^{t\alpha}\Gamma^{\nu}_{\alpha\nu} + \Gamma^{t}_{\alpha\nu}T^{\alpha\nu} \\
&= T^{tt}{}_{,t} + T^{tt}\Gamma^{\nu}_{t\nu} + \Gamma^{t}_{tt}T^{tt} + \sum_{j=1}^{3}\Gamma^{t}_{jj}T^{jj}
\end{aligned}$$

(where I have used the fact that $T$ is diagonal). Now recall an identity (6.40) for the summed Christoffel symbols:

$$\Gamma^{\alpha}_{\mu\alpha} = (\sqrt{-g})_{,\mu}/\sqrt{-g}.$$

This lets us calculate

$$
\begin{aligned}
\Gamma^{\nu}_{t\nu} &= (\sqrt{-g})_t/\sqrt{-g} \\
&= \frac{\partial}{\partial t}\ \ln(\sqrt{-g}) \\
&= \frac{1}{2}\,\frac{\partial}{\partial t}\ \ln(-g) \\
&= \frac{1}{2}\,\frac{\partial}{\partial t}\ \ln[A^6(1-kr)^{-1}r^4\cdot(\text{fn. of angles})] \\
&= \frac{1}{2}\cdot 6A^5A'\cdot A^{-6} \\
&= 3\,\frac{A'}{A}\,,
\end{aligned}
$$

$$\begin{aligned}\Gamma^t_{tt} &= \frac{1}{2}\,g^{tt}(g_{tt,t} + g_{tt,t} - g_{tt,t}) \\ &= -\frac{1}{2}\,g_{tt,t} \\ &= 0,\end{aligned}$$

and if $j$ is a spatial index,

$$\begin{aligned}\Gamma^t_{jj} &= \frac{1}{2}\,g^{tt}(g_{tj,j} + g_{jt,j} - g_{jj,t}) \\ &= \frac{1}{2}\,g_{jj,t} \\ &= \frac{A'}{A}\,g_{jj}\,.\end{aligned}$$

Therefore,

$$\sum_{j=1}^{3} \Gamma^t_{jj} g^{jj} = 3\,\frac{A'}{A}\,.$$

So the equation becomes

$$\begin{aligned} 0 &= \frac{d\rho}{dt} + 3\,\frac{A'}{A}\,\rho + 3\,\frac{A'}{A}\,p \\ &= \frac{1}{A^3}\left[\frac{d}{dt}(\rho A^3) + p\,\frac{d}{dt}(A^3)\right], \end{aligned}$$

which immediately implies the assertion.

8. One uses this conservation law and an equation of state to eliminate $p$. (For now, assume the cosmological constant $\Lambda$ is zero.)

   (A) For "cold" matter (or "heavy" particles), $p = 0$ and hence $\rho A^3 =$ constant.

   (B) At the opposite extreme, where all the matter is radiation (photons or neutrinos) *or* so hot that the rest masses of the particles can be neglected, we have $p = \frac{1}{3}\rho$ and hence (after a step of calculus) $\rho A^4 =$ constant.

Thus we have the famous prediction that as the universe expands, the density of matter falls off as $A^{-3}$ but that of radiation falls off as $A^{-4}$. (The latter can be explained as follows: Not only does the volume grow cubically, so that the number density of photons goes as the size to the $-3$ power, but also the wavelength of a photon grows linearly, cutting another power of $A$ from the energy density.)

9. Solve the Einstein equation (12.54) (with $\rho_\Lambda = 0$), getting the famous decelerating expansion from a Big Bang start. (Of course, there are also contracting solutions.)

10. Relate the expansion to observable quantities, the Hubble constant and the deceleration $q$ (Sec. 12.23). (For more detail on this (and other steps) than Schutz and I can provide, see the book of M. Berry, *Principles of Cosmology*

*and Gravitation.*)

11. Mix in some particle physics and statistical mechanics to relate the expansion to thermal background radiation, decoupling of matter from radiation, chemical element production, galaxy formation, etc. (Sec. 12.4).

12. Worry about what happens at very early times when the classical theory of matter (and of gravity?) must break down. As we go backwards, we expect to encounter

    a) phase transitions in field theories of elementary particles; inflation of the universe;

    b) creation of particles by the gravitational field (Parker et al.);

    c) quantum gravity; superstrings; ???.

At the cost of some repetition, I shall now go over RW cosmology in greater generality (allowing a cosmological constant and more general matter sources) and with a more systematic presentation of the starting equations.

The Einstein equation with $\Lambda$ term is

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}\,.$$

Its 00 component is

$$\left(\frac{\dot{A}}{A}\right)^2 + \frac{k}{A^2} - \frac{\Lambda}{3} = \frac{8\pi G}{3}\,\rho, \tag{1}$$

and all the $jj$ components are equivalent to

$$\frac{2\ddot{A}}{A} + \left(\frac{\dot{A}}{A}\right)^2 + \frac{k}{A^2} - \Lambda = 8\pi G p. \tag{2}$$

We also know that the only nontrivial component of the conservation law, $\nabla_\mu T^{\mu\nu} = 0$, is

$$\frac{d}{dt}(\rho A^3) = -p\,\frac{d}{dt}A^3. \tag{3}$$

The first task is to show that these 3 equations are not independent.

Differentiate (1):

$$\frac{2\dot{A}}{A}\left[\frac{\ddot{A}}{A} - \left(\frac{\dot{A}}{A}\right)^2\right] - \frac{2k}{A^2}\,\frac{\dot{A}}{A} = \frac{8\pi}{3}\,\dot{\rho}.$$

But (3) is equivalent to

$$\dot{\rho}A^3 + 3\rho A^2\dot{A} = -3pA^2\dot{A}, \quad \text{or} \quad \dot{\rho} = -3\,\frac{\dot{A}}{A}(\rho + p).$$

So we get

$$\frac{2\dot{A}}{A}\left[\frac{\ddot{A}}{A}-\left(\frac{\dot{A}}{A}\right)^2-\frac{k}{A^2}\right]=-8\pi\,\frac{\dot{A}}{A}(\rho+p).$$

Therefore, *either* $\dot{A}=0$ (a special situation to which we'll come back later) *or*

$$\frac{\ddot{A}}{A}-\left(\frac{\dot{A}}{A}\right)^2-\frac{k}{A^2}=-4\pi(\rho+p). \qquad (4)$$

Eliminate $p$ from (2) and (4): $\frac{1}{2}(2)-(4)$ is

$$\frac{3}{2}\left(\frac{\dot{A}}{A}\right)^2+\frac{3}{2}\,\frac{k}{A^2}-\frac{\Lambda}{2}=4\pi\rho.$$

But this is just (1) again, so (2) adds nothing to (1) and (3). (1) is effectively a *first integral* of (2) (which adds information corresponding to an integration constant). One notes that (3) is a bit nicer than (2) (simpler and more general), so it is standard to adopt (1) and (3) as the basic equations. However, we also need some information about the physics of the matter, in the form of an *equation of state*.

So, let's start over with three ingredients:

1. Equation of state,
$$p = f(\rho)$$
for some function $f$. (It tells how $p$ depends on $\rho$.)

2. Conservation law,
$$\frac{d}{dt}(\rho A^3) = -p\,\frac{d}{dt}A^3.$$

(It tells how $\rho$ depends on $A$.)

3. Einstein equation,

$$\left(\frac{\dot{A}}{A}\right)^2 + \frac{k}{A^2} - \frac{\Lambda}{3} = \frac{8\pi G}{3}\,\rho.$$

(It tells how $A$ depends on $t$.)

Thus we have a closed system of equations to tell how everything depends on $t$.

To start the solutiom. substitute the state equation into the conservation law:

$$\frac{d}{dt}(\rho A^3) = -f(\rho)\,\frac{d}{dt}A^3.$$

Define $u = A^3$ and use $u$ as the independent variable, so that

$$\frac{d}{dt} = \frac{du}{dt}\,\frac{d}{du}\,.$$

(This makes sense during any epoch through which $A(t)$ is monotonic.) We find

$$\frac{du}{dt}\,\frac{d}{du}(\rho u) = f(\rho)\,\frac{du}{dt}\,,$$

and after a few more steps of algebra,

$$\frac{d\rho}{du} = -\,\frac{\rho + f(\rho)}{u}\,.$$

This is an elementary first-order separable ODE, with solution

$$-\ln u + K = \int \frac{d\rho}{\rho + f(\rho)}\,. \tag{5}$$

To go further one needs a definite equation of state. Suppose that it has the special form

$$f(\rho) = w\rho. \tag{6}$$

Then

$$-\ln u + K = \frac{\ln \rho}{1+w}\,,$$

or

$$\rho = e^{(1+w)(-\ln u + K)} = Cu^{-(1+w)}\,,$$

or

$$\rho = Ca^{-3(1+w)}. \tag{7}$$

Now consider various particular cases of (6):

- Radiation (massless particles): $w = \frac{1}{3}$ — that is,

$$T^\alpha{}_\beta = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & \frac{\rho}{3} & 0 & 0 \\ 0 & 0 & \frac{\rho}{3} & 0 \\ 0 & 0 & 0 & \frac{\rho}{3} \end{pmatrix}.$$

(Note that the trace $T^\alpha_\alpha$ equals 0 in this case.) According to (7),

$$\rho \propto A^{-4}.$$

- Dust (very massive particles): $w = 0$, $p = 0$. In this case

$$\rho \propto A^{-3}.$$

  The physical reason for the difference in behavior is that for dust, the energy is essentially just the (conserved) mass, so the energy density is inversely proportional to the volume, whereas for photons the wavelength is stretching out, causing an additional factor of $A^{-1}$.

- The curvature term, $\frac{k}{A^2}$, acts like a type of fictitious matter with $w = -\frac{1}{3}$.

- The cosmological term, $-\frac{\Lambda}{3}$, acts like a type of fictitious matter with $w = -1$ (that is, $T_{\alpha\beta} = \frac{\Lambda}{8\pi G}\, g_{\alpha\beta}$, which is independent of $t$). In the observationalist's cautious approach to cosmological acceleration, one says that the *dark energy* has a stress tensor with $p = w\rho$ where $w \approx -1$. (The mathematician should be equally cautious and note that lots of our intermediate steps are nonsense when $1 + w = 0$; but you can go back and see that the starting equation (before (5)) is still satisfied.)

Accordingly, we shall now move the curvature and cosmological terms (if they are present) to the right-hand side of the Einstein equation and treat them mathematically as types of matter.

We can now easily solve Einstein in the *special case* where only one type of matter is present (or at least one type dominates).

$$\left(\frac{\dot{A}}{A}\right)^2 = \frac{8\pi G}{3}\rho = CA^{-3(1+w)}.$$

Therefore, $\dot{A} = \sqrt{C}\, A^{-(1+3w)/2}$, so

$$\sqrt{C}\, t + K = \int A^{\frac{1+3w}{2}}\, dA = \frac{2}{3(1+w)}\, A^{\frac{3(1+w)}{2}},$$

or

$$A^{\frac{3(1+w)}{2}} = \gamma t + \kappa.$$

We can choose the origin of time so that the arbitrary constant $\kappa$ is 0. Thus (for a new constant $C$)

$$A = (\gamma t)^{\frac{2}{3}\,\frac{1}{1+w}} = C t^{\frac{2}{3(1+w)}}. \tag{8}$$

Let's examine this result for the four classic cases.

- Radiation (with flat 3-space, $k = 0$): $w = \frac{1}{3}$, $A \propto t^{1/2}$.
- Dust (with $k = 0$): $w = 0$, $A \propto t^{2/3}$.

- Negative curvature ($k = -1$): $w = -\frac{1}{3}$, $A \propto t$. This solution, called the *Milne universe*, is actually a piece of *flat* space-*time* in hyperbolic coordinates.

- Cosmological constant: $w = -1$. Formula (8) fails, but we can go back to an earlier equation to see that $\dot{A} = \sqrt{C}\, A$, hence ($H \equiv \sqrt{C}$)

$$A = e^{Ht}.$$

  This solution is called the *de Sitter universe* (or a piece of it) with *Hubble constant* $H$.

If the universe contains more than one type of matter, or the equation of state is nonlinear, it is harder, perhaps impossible, to find exact solutions. Of course, that is no obstacle in serious research, since systems of ODEs are routinely solved by computer. (The fact that $A$ can vary over many orders of magnitude from the big bang to today means that the numerical analysis is not entirely trivial.) Much can be learned, as in elementary classical mechanics, by deducing the qualitative behavior of the solution. In general the Einstein equation will have the form

$$\left(\frac{\dot{A}}{A}\right)^2 = \rho(A),$$

where $\rho$ is typically a sum of terms corresponding to matter of various types (including curvature and $\Lambda$). This equation is quite analogous to the classical energy law for a particle in a potential,

$$\tfrac{1}{2}\dot{A}^2 = E - V(A),$$

for which we know how to classify solutions as bouncing back and forth in a potential well, passing over a potential barrier, etc.

Usually *one* term in $\rho$ is dominant in any particular epoch, and the solution can be approximated by one of the power-law behaviors we found earlier. Let $t = 0$ at the initial singularity. When $t$ is very small, matter (in particular, radiation) dominates over spatial curvature, so one has a $t^{1/2}$ model. (When a $k = 1$ curvature term is included, the exact solution is a cycloid, which clearly demonstrates the $t^{1/2}$ behavior at the start.) Because radiation density goes as $A^{-4}$ and heavy matter density goes as $A^{-3}$, matter eventually comes to dominate, and we have a period with $A \propto t^{2/3}$. Still later the spatial curvature becomes important and can make the difference between a universe that expands forever and one that recollapses. For ordinary matter one must have $\rho > 0$ and $0 \le p \le \rho$. If $\Lambda = 0$, one can then show that recollapse occurs if and only if $k > 0$. Finally, if there is a cosmological constant and the universe becomes large enough, the $\Lambda$ term will eventually win out over all the others, presumably producing the accelerated expansion that we now observe.

I promised to say something about what happens if $\dot{A} = 0$. We are not concerned here with a case where one of the generic solutions has $\dot{A} = 0$ at an isolated time; the solution can be extended through such a point (usually changing from expanding to contracting) even though our derivation technically breaks down there. Are there solutions with $A$ equal to a constant? One quickly sees that that would place constraints on $\rho$, $p$, $k$, and $\Lambda$. Most of the solutions are either flat space-time or otherwise uninteresting. The most interesting one is called the *Einstein universe*: It has

$$\rho = \frac{C}{A^3} = \frac{1}{4\pi G A^2}\,, \quad p = 0, \quad k = 1, \quad \Lambda = \frac{1}{A^2}\,.$$

(See Schutz, Exercise 12.20.) In the early days of general relativity, it was taken very seriously ("Einstein's greatest blunder").

## The present observational situation

At this point in the original notes I recommended and summarized a review article, "The case for the relativistic hot big bang cosmology", by P. J. E. Peebles, D. N. Schramm, E. L. Turner, and R. G. Kron, *Nature* **352**, 769–776

(1991). At the time, it was more up to date than the first edition of Schutz's book; of course, that is no longer true, and Secs. 12.2 and 12.4 provide some newer information. Since 1991 there has been considerable progress, notably the firm discovery of the acceleration of the universe, which provides evidence for a nonzero cosmological constant (or, more cautiously, "existence of dark energy"). For the state of the art after 5 years of analysis of WMAP data (2008), see `http://www.math.tamu.edu/~fulling/WMAP/html` on our Web page.

# Spherical Solutions: Stars and Black Holes *(Chapters 10 and 11)*

We will seek solutions ($g_{\mu\nu}$ or $ds^2$) of Einstein's equation that are

1. **static**:

   a. time-independent (in some coordinate system);

   b. time-reversal invariant (no $dt\,dx^j$ terms in $ds^2$).

2. **spherically symmetric**: angular part of $ds^2$ is

$$r^2\,d\Omega^2 \equiv r^2(d\theta^2 + \sin^2\theta\,d\phi^2).$$

3. **asymptotically flat**: the geometry in some sense approaches that of flat space-time as $r \to \infty$.

**Remarks:**

(1) Here we have defined $r$ so that $2\pi r$ is the circumference of the spherical surface at fixed $r$. When the space is not flat, $r$ is not necessarily a radial proper distance — that is, the $dr^2$ term will turn out to be $f(r)\,dr^2$ with some nontrivial function $f$. You could equally well *define* a proper radial coordinate $\rho$ by

$$d\rho^2 = f(r)\,dr^2\,; \qquad \rho \equiv \int f(r)^{1/2}\,dr,$$

but then the $r^2$ multiplying $d\Omega^2$ would be a nontrivial function of $\rho$. (You might want to make $\rho$ unique by choosing the lower limit of the integral to be 0 (so "$\rho$ is the distance from the center"); but we should leave open the possibility that there is no point where $r = 0$ — in fact, something like that happens for a black hole.)

(2) A space-time that satisfies 1a but not 1b is called *stationary*. Systems that are *rotating* at constant angular velocity are of that type. (It turns out that they also violate the spherical symmetry condition, being symmetrical about the axis of rotation only.)

The most general metric satisfying the three symmetry conditions is

$$ds^2 = -e^{2\Phi(r)}\,dt^2 + e^{2\Lambda(r)}\,dr^2 + r^2\,d\Omega^2.$$

(The exponential notation incorporates what we know about the signs of the $g_{\mu\nu}$ and turns out to simplify the equations of motion somewhat. The factor 2 is natural because, as we just saw, there is often a reason to take square roots of metric components.)

$\Phi(r)$ has an important observational significance. If a photon is emitted at radius $r$ and observed far away, its frequency will suffer a *red shift*

$$\frac{\lambda_{\text{received}} - \lambda_{\text{emitted}}}{\lambda_{\text{emitted}}} \equiv z = -g^{00}(r) + g^{00}(\infty) = e^{-\Phi(r)} - 1.$$

The reasoning is essentially the same as in Sec. 5.1 (Schutz says "Chapter 2" but he doesn't mean it).

The components of the Einstein tensor for such a metric have been calculated in (10.14–17) (and homework). We also have the conservation law (Bianchi identity), (10.27),

$$(\rho + p)\,\frac{d\Phi}{dr} = -\,\frac{dp}{dr}\,,$$

which has the local physical interpretation of hydrostatic pressure balance in the star. As in the cosmological theory, it pays to substitute this equation for one of the Einstein equations, say the one for $G_{\theta\theta}$ .

## The exterior Schwarzschild solution

Assume for now that $T_{\mu\nu} = 0$, as it should be (approximately) outside a star (at least until we run into another star). Then the four Einstein equations are

$$\begin{aligned}
0 = G_{00} &= \frac{1}{r^2}\, e^{2\Phi}\, \frac{d}{dr}\big[r\big(1-e^{-2\Lambda}\big)\big], \\
0 = G_{rr} &= -\,\frac{1}{r^2}\, e^{2\Lambda}\big(1-e^{-2\Lambda}\big) + \frac{2}{r}\,\Phi', \\
0 = G_{\theta\theta} &= r^2 e^{-2\Lambda}\left[\Phi'' + (\Phi')^2 + \frac{\Phi'}{r} - \Phi'\Lambda' - \frac{\Lambda'}{r}\right], \\
0 = G_{\phi\phi} &= \sin^2\theta\, G_{\theta\theta}\,.
\end{aligned}$$

Obviously the $\phi\phi$ equation is redundant. In fact, the $\theta\theta$ equation itself will be automatic (Bianchi identity!), because the conservation law is tautologically satisfied in vacuum.

The trick to solving the two remaining equations is to define

$$m(r) = \frac{r}{2}\big(1-e^{-2\Lambda}\big).$$

(It will turn out that $m(r)$ can be thought of as the mass inside the ball of radius $r$, but not in the usual sense of a straighforward integral of $\rho$.) Inverting this equation gives

$$g_{rr}(r) = e^{2\Lambda(r)} = \left(1 - \frac{2m(r)}{r}\right)^{-1}.$$

But now the content of the 00 Einstein equation is that $m(r) = \text{constant}$ in the vacuum region! Furthermore, the $rr$ equation can be written

$$\frac{d\Phi}{dr} = \frac{m}{r(r-2m)}.$$

Its solution that vanishes at infinity is

$$\Phi(r) = \frac{1}{2}\ln\left(1 - \frac{2m}{r}\right), \quad \text{or} \quad -g_{00}(r) = e^{2\Phi(r)} = 1 - \frac{2m}{r}.$$

Thus, finally, we have arrived at the famous *Schwarzschild solution*,

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2\, d\Omega^2.$$

Almost equally famous is –

**Birkhoff's theorem:** The only spherically symmetric, asymptotically flat, vacuum solution is the Schwarzschild solution.

Note that the *static* condition is not necessary as a hypothesis in Birkhoff's theorem! Even if something wild is going on inside a star, the gravitational field outside is still Schwarzschild, provided that the phenomenon is spherically symmetric. An equivalent statement is that *there is no monopole gravitational radiation*. The same is true of electomagnetic radiation: A radially pulsating electric charge distribution has nothing but a Coulomb field outside. The most basic electromagnetic radiation is *dipole* (spherical harmonics with $l = 1$), corresponding to opposite charges with varying separation (or a single charge with oscillating position). Since all masses are positive, even that possibility does not exist for gravitational waves. The most basic gravitational radiation is *quadrupole* (spherical harmonics with $l = 2$), corresponding to a change of shape of a matter distribution (say from prolate to oblate). (See Sec. 9.3; also Figs. 9.1 and 9.2, but interpreted as referring to the source, not the detector.)

## Inside a star

Recall that $T_{\mu\nu} = pg_{\mu\nu} + (\rho + p)U_\mu U\nu$, where $U$ is the unit vector in the time direction, so $T_{00} = -\rho g_{00}$. Thus

$$T_{00} = \rho e^{2\Phi}, \quad t_{rr} = pe^{2\Lambda}, \quad T_{\theta\tau} = r^2 p, \quad T_{\phi\phi} = r^2 \sin^2\theta\, p.$$

Returning to the Einstein equations, therefore, one sees that the 00 equation amounts to

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho, \tag{A}$$

and the $rr$ equation to

$$\frac{d\Phi(r)}{dr} = \frac{m(r) + \pi r^3 p(r)}{r(r - 2m(r))} \ .$$

But $d\Phi/dr$ also appears in the conservation law, which can be used to rewrite the $rr$ equation as the *Tolman–Oppenheimer-Volkov equation*,

$$\frac{dp}{dr} = -(\rho + p)\frac{m + 4\pi r^3 p}{r(r - 2m)} \ . \tag{B}$$

As usual, one needs to assume an equation of state,

$$p = p(\rho) \qquad \text{(C).}$$

The system (A)–(B)–(C) has a unique solution once two constants of integration are specified. One of these is the initial condition $m(0) = 0$, without which the solution would be singular at the origin (see p. 264 for complete argument). The other is the central pressure, $p(0)$ (or, alternatively, the central density, $\rho(0)$). Thus, for a given equation of state, the spherically symmetric static stellar models form a one-parameter sequence. Usually the solution can't be written down analytically, but finding it is a standard project in numerical analysis.

## The Schwarzschild horizon *(Section 11.2)*

Now we return to the vacuum Schwarzschild solution and ask what would happen if the "star" is so small that the factor $1 - 2m/r$ can become zero in the region where that solution applies. (We celebrate its importance by changing $m$ to a capital $M$.) At that point ($r = 2M$) the metric expression is singular (at least in the coordinates we're using, and if $r$ becomes smaller than $2M$, the signs of the time and radial metric components change. We must ask whether it makes any sense to continue the solution inward in this way. The answer is "yes", but the metric expression needs to be interpreted carefully.

Let us distinguish three physical situations:

1. An ordinary star, but sufficiently dense that general relativity is significant (e.g., a neutron star). Then as explained in Chapter 10, the *exterior Schwarzschild solution*

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

applies in the region where there is no matter. This region necessarily ends when $r$ goes below some minimum value greater than $2M$. Inside that radius the geometry is described by some other spherically symmetrical metric, determined by Einstein's equation with a matter source. (Sometimes that metric is called the *interior Schwarzschild solution*.)

2. The *maximal analytic extension* of the Schwarzschild metric up to and through the coordinate surface $r = 2M$. This describes an empty space that, at a fixed time, resembles the "wormhole" drawing in Schutz Fig. 10.1 (which was on the cover of the first edition of the book). That is, there is a "second sheet" of the universe, where $r$ begins to increase again. Moreover, in the region $r < 2M$, $r$ and $t$ exchange roles, so that $r$ is the time coordinate. In this region the geometry is not time-independent: The "neck" of the "wormhole" grows from nothing and then shrinks to nothing within a finite proper time. There is a genuine singularity at $r = 0$, where the curvature becomes infinite and the manifold cannot be extended. (In fact, there are two of these, one in the future and one in the past.) However, there is no true singularity at $r = 2M$. (That surface is highly interesting, nonetheless; it is called the *horizon* and will be discussed extensively hereafter.) Such a

maximal black hole is not expected to exist in the real world (except possibly at the quantum scale, where it would presumably be modified).

3. A collapsing (later, collapsed) star, such as is expected to be the fate of a burned-out star too massive to become a white dwarf or neutron star. At early times, this resembles a static star. At late times, it resembles part of the maximal extension. The Schwarzschild metric is applicable only outside the collapsing matter, but the latter eventually shrinks below the radius $2M$. This empty Schwarzschild region does *not* include the second sheet of the wormhole, nor the past singularity, but it *does* include a part of the region extending from the horizon ($r = 2M$) to the future singularity ($r = 0$). This scenario is the *black hole* of the realistic astrophysicist; such things may actually exist and cause observable phenomena.

What exactly is happening at $r = 2M$? Detailed investigation reveals the following about the Schwarzschild coordinate system:

1. As $r \to \infty$, the metric of course approaches that of flat (Minkowski) space in spherical coordinates. In that limit, therefore, the coordinate system can be thought of as a standard inertial coordinate system in polar form.

2. As $r \to 2M$, however, these coordinates have more in common with the hyperbolic polar coordinates that we have discussed several times in connection with uniform acceleration. (The reason for this analogy is clear: Near a massive body, a point *at rest* at a constant radius *is* accelerating; if it were in free fall, its $r$ would be decreasing!) After the two angular coordinates are suppressed, the coordinate $r$ measures the distance of a point from a single central point. (When the angles are restored, the central point, like every normal point in the $(r,t)$ plane, represents an entire sphere in the 4-dimensional manifold. Here "normal" excludes $r \leq 0$ and $r = 2M$.) Translation in $t$ represents a motion around that point like a Lorentz transformation, rather than a conventional time translation.

In hyperbolic coordinates in Minkowski space, there is a coordinate singularity at $\chi = 0$ just like the one at $r = 2M$ here. The singularity can be removed by returning to Cartesian coordinates:

$$t = r \sinh \chi,$$
$$x = r \cosh \chi.$$

(We can also introduce hyperbolic coordinates inside the future light cone by

$$t = r \cosh \chi,$$
$$x = r \sinh \chi.$$

This is the analogue of the region $0 < r < 2M$ in the black hole.) Analogous coordinates should exist in the case of the black hole (including the case of the collapsed star, outside the collapsing matter). They are called *Kruskal coordinates.* Although they are analogous to Cartesian coordinates in Minkowski space, they are *not* adapted to symmetries of the metric (other than rotations): the geometry is *not* invariant under translation in Kruskal time. The coordinate transformation is given in Schutz (11.65–66) and the resulting line element in (11.67); note that the latter cannot be written explicitly in terms of the Kruskal coordinates themselves, but only implicitly through $r$.

Although Kruskal coordinates are hard to work with algebraically, they are easy to understand geometrically if we introduce some intermediate coordinate systems called *null coordinates*. Note first that we are analogizing the radius-time plane in the black hole space-time to the 2-dimensional Minkowski space — suppressing the angles in the one case and two transverse dimensions in the other. In Minkowski space with coordinates $(x, t)$, let's let

$$U = x - t, \qquad V = x + t.$$

It is well-known that this converts the two-dimensional wave operator $\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}$ into a multiple of $\frac{\partial^2}{\partial U\,\partial V}$, which is easily solved (*d'Alembert's solution*). In semi-

Riemannian geometry terms, it converts the line element $ds^2 = -dt^2 + dx^2$ into $dU\,dV$. The lines of constant $U$ or constant $V$ are diagonal lines on the $(x, t)$ diagram; they are photon paths! Thus such a coordinate system exhibits the physical geometry of a (effectively two-dimensional) space-time very directly.

Now consider, in the quadrant $U > 0$, $V > 0$, the further transformation

$$U = e^u, \qquad V = e^v.$$

Then $ds^2 = e^{u+v}\,du\,dv$, so the lines of constant coordinate are still the same null lines, just labelled differently. Next, rediagonalize by reversing the Cartesian-to-null transformation:

$$u = \xi - \tau, \qquad v = \xi + \tau.$$

$$ds^2 = e^{2\xi}(-d\tau^2 + d\xi^2).$$

Finally, let

$$\xi = \ln r, \qquad \tau = \chi.$$

Composing all the transformations, we see that

$$x = \frac{1}{2}(U+V) = \frac{1}{2}(e^u + e^v) = \frac{1}{2}e^\xi(e^{-t} + e^t) = r\cosh\chi,$$

$$t = r\sinh\chi.$$

So we have recovered the hyperbolic coordinates. Incidently, the metric in that system is

$$ds^2 = -r^2\,dt^2 + dr^2,$$

which looks vaguely Schwarzschildian, does it not?

The relation between Schwarzschild and Kruskal coordinates is identical to this, except for the function relating $r$ to $\xi$. (The coordinate analogous to $\xi$ is called the *tortoise coordinate* because it sends the value $r = 0$ to $\xi = -\infty$, making it appear to take infinitely long to reach, as in Zeno's paradox. More about this below.) Similar coordinate transformations appear in other contexts, notably in de Sitter space when one follows a natural static coordinate system "quarter way around the world". Note that Schutz uses $(u, v)$ for the analogue of $(x, t)$, not for the coordinates that I (following most relativists) have called $(u, v)$ (and particle theorists call $(x_-, x_+)$, up to a factor $\sqrt{2}$).

It is also possible to perform nonlinear transformations on null coordinates so as to map "infinity" into finite null lines. The fictitious light cones at infinity introduced in this way are called $\mathcal{I}^+$ and $\mathcal{I}^-$. In studying the wave or field equations of massless particles, it is convenient to prescribe initial data there.

It is important to make the distinction between the line $r = 2M$ in a Schwarzschild diagram (Fig. 11.10) and the line $r = 2M$, $t = +\infty$ in a Kruskal diagram (Fig. 11.11). The former line shrinks to only *one point* on the Kruskal diagram. The latter line is the (future) *horizon*. It is crucial to understand that locally there is nothing out of the ordinary about the geometry or the physics there, at least if $M$ is large so that the curvature is still fairly small at $r = 2M$.

Generally speaking, an horizon is defined only *globally*, in terms of its relationship to a singularity in the future. It is the boundary between points from which one can escape to infinity ($\mathcal{I}^+$) without exceeding the speed of light, and points from which one is doomed to fall into the singularity instead. Incidentally, the Schwarzschild singularity is anisotropic: the tidal forces are compressive in two dimensions and stretching in the third (along the singularity, drawn as a spacelike hyperbolic curve on the Kruskal diagram). A later infall of matter may relocate the horizon without changing the geometry "here and now".

The formula for the tortoise coordinate is

$$r^* \equiv \int \left(1 - \frac{2M}{r}\right)^{-1} dr = r + 2M \ln\left(\frac{r}{2M} - 1\right).$$

Then

$$ds^2 = \left(1 - \frac{2M}{r}\right)(-dt^2 + dr^{*2}) + \cdots,$$

where $r$ is a function of $r^*$ (implicitly defined as the inverse of the previous formula). If we are looking at the space-time outside a collapsing star, whose surface $r(t)$ crosses the horizon at some finite Kruskal time, then it can be shown that for large $t$ the path of the surface looks asymptotically straight and diagonal on the $(r^*, t)$ diagram; more precisely,

$$r^* \sim -t + Ae^{-t/2M} + B \sim -\ln\left(\cosh\frac{t}{2M}\right).$$

For Schwarzschild and Kruskal diagrams in the scenario of matter collapsing to a black hole, see Fig. 4 of Davies and Fulling, *Proc. Roy. Soc. London A* **356**, 237–257 (1977).

## The Ergosphere *(Section 11.3)*

A rotating (Kerr) black hole in the most common (Boyer–Lindquist) coordinate system has a metric of the form

$$ds^2 = -(\text{mess})dt^2 - (\text{mess})dt\,d\phi + (\text{mess})d\phi^2 + (\text{mess})dr^2 + (\text{mess})d\theta^2$$

(see Schutz (11.71) and Penrose–Floyd p. 2). The (mess)s are independent of $t$ and $\phi$, so the energy, $-p_t$, and angular momentum, $p_\phi$, of a particle are conserved. (Recall that $p_0$ is normally a *negative* number in our metric signature, since $U^0$ is positive.)

Suppose the hole were replaced by a spinning flywheel. A particle could hit it and be batted away with more energy than it had coming in. This does not contradict conservation of energy, because there is a nontrivial interaction with the flywheel and the wheel will slow down slightly by recoil. The Penrose process is an analog that allows energy to be extracted from the Kerr black hole.

As in the Brans–Stewart cylinder universe, there is no *global* rotating Lorentz frame. (This is true of any rotational situation in relativity — it has nothing to do with horizons or even with gravity.) The best one can do is to construct a rotating frame that is related to local Lorentz frames by *Galilean* transformations (i.e., leaving the hypersurfaces of constant time fixed).

*The model*

Here I present a simple model related to the Kerr black hole in somewhat the same way that the uniformly accelerated (Rindler) frame is related to the Schwarzschild black hole. Consider the line element

$$ds^2 = -dt^2 + (dx + V(y)\,dt)^2 + dy^2.$$

(We could add a third spatial dimension, $dz$, but it adds nothing conceptually so

I'll omit it.) That is, the metric tensor is

$$g_{\mu\nu} = \begin{pmatrix} -1+V(y)^2 & V(y) & 0 \\ V(y) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where the order of the coordinates is $t$, $x$, $y$. Since $g_{\mu\nu}$ is independent of $t$ and $x$, $-p_t$ and $p_x$ are conserved. Notice that something strange is going to happen when $|V(y)| \geq 1$, because then $g_{tt}$ changes sign.

Consider now the Galilean transformation

$$t = t', \qquad x = x' - V_0 t', \qquad y = y',$$

with inverse

$$t' = t, \qquad x' = x + V_0 t, \qquad y' = y.$$

Then $dx = dx' - V_0\,dt'$ implies

$$ds^2 = -dt'^2 + [dx' + (V(y) - V_0)dt']^2 + dy'^2.$$

In particular, in a region where $V(y) = \text{constant}$, choose $V_0 = V(y)$; then

$$ds^2 = -dt'^2 + dx'^2 + dy'^2$$

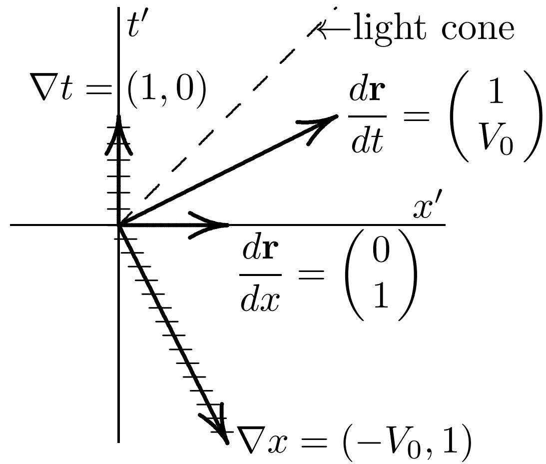
— space is flat!

Suppose that $V(y) = 0$ for $y \gg 0$ ("outside"), so the space is flat and the unprimed coordinates are inertial there; and that $V(y) = V_0$ for $y \ll 0$ ("inside"), so the space is flat and the primed coordinates are inertial there. In the Kerr–Boyer–Lindquist situation, $r$ is analogous to $y$ and $\phi$ is analogous to $x$. Like the Schwarzschild black hole, the Kerr black hole has a horizon at some small $r \equiv r_+$ (and a singularity inside that), but that does not concern us today. We are interested in a region $r_+ < r < r_0$ called the *ergosphere*. (See Schutz p. 312 for formulas for $r_+$ and $r_0$; $r_0$ is where $g_{tt} = 0$, and $r_+$ is where $g_{rr} = \infty$.) In our model, the ergosphere is the inside region, $-\infty < y \ll 0$.

Let us look at the unprimed basis vectors in primed terms; in other words, look at the (natural interior extension of the) inertial frame of an observer in the exterior region from the point of view of an observer "going with the flow" in the interior region. The change-of-basis matrices are

$$\Lambda^{\mu}{}_{\nu'} = \frac{\partial x^{\mu}}{\partial x^{\nu'}} = \begin{pmatrix} 1 & 0 & 0 \\ -V_0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \Lambda^{\nu'}{}_{\mu} = \frac{\partial x^{\nu'}}{\partial x^{\mu}} = \begin{pmatrix} 1 & 0 & 0 \\ V_0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Recall that the columns of the second matrix are the basis tangent vectors $\vec{e}_t$, etc., and the rows of the first matrix are the basis one-forms dual to them. The important thing to note is that if $V_0 > 1$, then $\vec{e}_t$, the time-translation vector, is spacelike in the ergosphere! (On the other hand, $\nabla t$, the normal vector to the surfaces of constant $t$, is still timelike.)




Similarly, in Kerr, $\vec{e}_t$ in the ergosphere leans over and points primarily in the $\phi$ direction. (In any rotating system in GR, it will lean slightly; this is called the *Lense–Thirring frame-dragging effect*, or *gravitomagnetism*; see Schutz pp.

310–311. But usually it remains timelike. An ergosphere is a region where it leans so far it becomes spacelike.)

*Velocity*

Let's use $\Lambda$ to transform the 4-velocity vector of a particle:

$$\vec{v} = \begin{pmatrix} 1 & 0 & 0 \\ -V_0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{v}' = \begin{pmatrix} v_{t'} \\ v_{x'} - V_0 v_{t'} \\ v_{y'} \end{pmatrix} .$$

Now suppose that the spatial velocity is 0 in the unprimed frame; then

$$v_{x'} = V_0 v_{t'} .$$

But if $|V_0| > 1$, this equation would say that $\vec{v}$ is spacelike, which is impossible for a physical particle. Conclusion: *A particle inside the ergosphere cannot be motionless as viewed by an observer outside.*

Because the metric is nondiagonal, the canonical momentum is not proportional to the velocity. The Lagrangian for particle motion is

$$L = \tfrac{1}{2}[(V(y)^2 - 1)\dot{t}^2 + 2V(y)\dot{t}\dot{x} + \dot{x}^2 + \dot{y}^2].$$

Therefore,

$$p_y = \frac{\partial L}{\partial \dot{y}} = \dot{y}, \qquad \frac{dp_y}{dt} = \frac{\partial L}{\partial y} = VV'\dot{t}^2 + V'\dot{t}\dot{x} = V'\dot{t}p_x\,,$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = V\dot{t} + \dot{x} = \gamma(V + v), \qquad \frac{dp_x}{dt} = \frac{\partial L}{\partial x} = 0,$$

$$p_t = \frac{\partial L}{\partial \dot{t}} = (V^2 - 1)\dot{t} + V\dot{x}, \qquad \frac{dp_t}{dt} = \frac{\partial L}{\partial t} = 0.$$

We can further reduce

$$p_t = -\dot{t} + V(V\dot{t} + \dot{x}) = -\dot{t} + V p_x \,.$$

Thus $\dot{t} = -p_t + V p_x$ , and we can write

$$\ddot{y} = \frac{dp_y}{dt} = V'(y)p_x[V(y)p_x - p_t],$$

which is the only nontrivial equation of motion. (Recall that $p_x$ and $p_t$ are constants.) Note that $p_y$ = constant whenever the particle is in either of the asymptotic regions.

*Energy extraction*

Consider a particle originating outside with

$$p_t = p_0 < 0, \qquad p_x = 0, \qquad p_y = -k < 0.$$

Since the inertial frame outside is the unprimed one, $p_t < 0$ is required for a physical particle. The condition $p_y < 0$ assures that the particle will fall in. In the primed frame these momentum components are the same:

$$\vec{p}\,' = (p_0, 0, p_y) \begin{pmatrix} 1 & 0 & 0 \\ -V_0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (p_0, 0, p_y) = \vec{p}.$$

In general, $p_y$ will change with time, but $p_x$ and $p_t$ are conserved. Let's say that $p_y = -K < 0$ when the particle is inside.

Now suppose that after it enters the ergosphere, the particle decays:

$$\vec{p} = \vec{p}_1 + \vec{p}_2\,.$$

(This is a vectorial equation, hence valid in either frame.) Suppose also that

$$p'_{2y} = +K > 0, \qquad \text{so} \quad p'_{1y} = -2K < 0.$$

Thus particle 1 gets swallowed by the "black hole", but particle 2 reemerges. In exterior coordinates

$$\vec{p}_1 = (p'_{1t}, p'_{1x}, p'_{1y}) \begin{pmatrix} 1 & 0 & 0 \\ +V_0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (p'_{1t} + V_0 p'_{1x}, p'_{1x}, p'_{1y}).$$

Note that $p_{1t} = p'_{1t} + V_0 p'_{1x}$ can be positive if (and only if) $|V_0| > 1$ (since $|p'_{1x}| < |p'_{1t}|$). (This is not a physical contradiction, since the unprimed frame is not inertial at points inside.) Now do the same calculation for the escaping particle:

$$\vec{p}_2 = (p'_{2t}, p'_{2x}, p'_{2y}) \begin{pmatrix} 1 & 0 & 0 \\ +V_0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (p'_{2t} + V_0 p'_{2x}, p'_{2x}, p'_{2y}).$$

Here $p_{2t} = p'_{2t} + V_0 p'_{2x}$ can be less than $p_0$ (i.e., $|p_{2t}| > |p_0|$) if and only if $|V_0| > 1$. But $p_{2t}$ is conserved, so it is the physical momentum of particle 2 after it emerges from the ergosphere.

Conclusion: *Mechanical energy has been extracted from the "black hole". Total energy is conserved, because the energy of the hole has been reduced by the amount* $|p_{1t}| = |p_0|$, *the negative energy carried in by particle 1.* In the true rotating-black-hole case, the angular momentum is reduced similarly (corresponding to the conserved quantity $p_x$ in the model).

There is an analogue of the Penrose process for waves, called *superradiance*: For waves of certain values of angular momentum (angular quantum number or separation constant), the scattered wave amplitude exceeds the incident amplitude. In quantum field theory this effect leads to production of particle-antiparticle pairs by the rotating black hole, in analogy to something called the *Klein paradox* for quantum particles in a strong electric field. (This is different from *Dicke superradiance* in atomic physics.)