Math 460

Problem 2.19 Solution Part (a): Considering a uniformly accellerated body, we must show that in the MCRF, the accelleration four-vector takes the form $\vec{a} = (0, \text{const.})$. Following the hint (problem 21), let us consider a body whose motion is parameterized

$$t(\tau) = \frac{1}{\alpha}\sinh(\alpha\tau), \qquad x(\tau) = \frac{1}{\alpha}\cosh(\alpha\tau) \tag{1}$$

Notice that

$$\vec{u} = \left(\frac{dt}{d\tau}, \frac{dx}{d\tau}\right) = (\cosh(\alpha\tau),\ \sinh(\alpha\tau) \tag{2}$$

so that

$$\vec{u}\cdot\vec{u} = -\cosh^2(\alpha\tau) + \sinh^2(\alpha\tau) = -1 \tag{3}$$

So τ is the proper time parameter. Computing the acceleration,

$$\vec{a} = \left(\frac{d^2t}{d\tau^2}, \frac{d^2x}{d\tau^2}\right) = (\alpha\sinh(\alpha\tau),\ \alpha\cosh(\alpha\tau)) \tag{4}$$

This has magnitude

$$\vec{a}\cdot\vec{a} = -\alpha^2\sinh^2(\alpha\tau) + \alpha^2\cosh^2(\alpha\tau) = \alpha^2 \tag{5}$$

confirming that this really is uniform acceleration. Now we need a Lorentz transformation to the MCRF. The coefficients for this Lorentz transformation can be read off from the components of $\vec{u}$:

$$\vec{u} = (\gamma,\ \gamma v) \tag{6}$$

Thus $\gamma = \cosh(\alpha\tau)$ and $\gamma v = \sinh(\alpha\tau)$. Now transforming $\vec{a}$:

$$\begin{aligned}\vec{a}_{MCRF} =&(\gamma a_t - \gamma v a_x,\ \gamma a_x - \gamma v a_t)\\ =&(\cosh(\alpha\tau)\alpha\sinh(\alpha\tau) - \sinh(\alpha\tau)\alpha\cosh(\alpha\tau),\\ &\cosh(\alpha\tau)\alpha\cosh(\alpha\tau) - \sinh(\alpha\tau)\alpha\sinh(\alpha\tau))\\ =&(0,\alpha)\end{aligned} \tag{7}$$

which is what needed to be shown. We may interpret α as Galilean acceleration by the following argument. At the moment $t = 0$, the lab frame coincides with the MCFR. At this time, we may compute $\frac{d^2x}{dt^2}|_{t=0}$ using the expression for $x(t)$ below, and find that it equals α.

Part (b): For the uniformly accelerated trajetcory above, we are asked to find the position as a function of t, the velocity as a function of t, and the time t needed to reach speed 0.999c. Notice that

$$x^2 - t^2 = 1/\alpha^2 \tag{8}$$

so we may immediately conclude

$$x(t) = \sqrt{\frac{1}{\alpha^2} + t^2} \tag{9}$$

and

$$v(t) = \frac{dx}{dt} = \frac{t}{\sqrt{\frac{1}{\alpha^2} + t^2}} \tag{10}$$

This equation may be solved for t as

$$t(v) = \frac{v}{\alpha\sqrt{1 - v^2}} \tag{11}$$

Setting $\alpha = 10$ and being careful here about units, we find

$$t(0.999c) = \frac{0.999c}{10\sqrt{1 - (0.999)^2}} = 6.7 \times 10^8\text{s} \tag{12}$$

Part (c): We are asked to find the elapsed proper time to reach .999c, and the elapsed proper time to travel a distance 2×10^{20}m. First, find $\tau(t)$ by integrating $d\tau$ (or alternatively, just invert equation (1)):

$$\begin{aligned} \tau(t) &= \int_0^t d\tau = \int_0^t \sqrt{1 - v(t)^2} dt = \int_0^t \sqrt{1 - \frac{t^2}{\frac{1}{\alpha^2} + t^2}}\, dt \\ &= \frac{1}{\alpha} \ln\left(\alpha t + \sqrt{1 + (\alpha t)^2}\right) \end{aligned} \tag{13}$$

Using the value of t from part (b), we find the proper time to reach 0.999c is 1.14×10^8s, or 3.6 years.

To travel a distance of 2×10^{20}m, notice that our starting x-coordinate is actually $x(0) = 1/\alpha$, which in appropriate units is 9×10^{15}m. However, for distances much larger than this,

$$t(x) = \sqrt{x^2 - \frac{1}{\alpha^2}} \approx x \tag{14}$$

So let us use this approximation and convert units to get the time coordinate

$$t \approx 6.67 \times 10^{11}\text{s} \tag{15}$$

and then use equation (13) to find the proper time:

$$\tau = 3.21 \times 10^{8}\text{s} = 10.2 \text{ years} \tag{16}$$