Math. 460 (Fulling) 7 October 2019

Midterm Test – Solutions

1. *(40 pts.)*

(a) Explain in modern language (multilinear functionals and all that) what a $\binom{1}{1}$ tensor is. (Call the tensor T.)

[This is on the "essay question" borderline, so I won't provide a model answer. But you should define "multilinear" (or, better, "bilinear").]

(b) Suppose that we are working in Minkowski space (the standard 4-dimensional flat space-time). How do the components of T change under a Lorentz transformation?

To make the notation precise: Let $\Lambda^{\overline{\alpha}}{}_{\beta}$ map the coordinates (t, x, y, z) into the coordinates $(\overline{t}, \overline{x}, \overline{y}, \overline{z})$. Write the new tensor components $\{T^{\overline{\mu}}{}_{\overline{\nu}}\}$ in terms of the old ones, $\{T^{\kappa}{}_{\lambda}\}$.

$$T^{\overline{\mu}}{}_{\overline{\nu}} = \Lambda^{\overline{\mu}}{}_{\kappa}\Lambda^{\lambda}{}_{\overline{\nu}}T^{\kappa}{}_{\lambda}\,.$$

Here $\Lambda^{\lambda}{}_{\overline{\nu}}$ is the matrix of the coordinate transformation in the opposite direction. Note that equivalent, more abstract formulas are

$$\overline{T} = \Lambda T \Lambda^{-1} \qquad \text{(similarity transformation of matrices)}$$

and

$$\overline{T} = \Lambda(\Lambda^{-1})^{\mathrm{t}}T$$

(if you think of the transformation matrices as acting sequentially on the two indices of the tensor T).

(c) The *trace* of T is the number $T^{\mu}{}_{\mu}$ (summation convention in force). Show that the trace is the same, no matter what coordinate system (or basis) is used.

$$T^{\overline{\mu}}{}_{\overline{\mu}} = \Lambda^{\overline{\mu}}{}_{\kappa}\Lambda^{\lambda}{}_{\overline{\mu}}T^{\kappa}{}_{\lambda} = \delta^{\lambda}_{\kappa}T^{\kappa}{}_{\lambda} = T^{\kappa}{}_{\kappa}\,.$$

(d) Does it matter whether the transformation in (c) is a Lorentz transformation (rather than an arbitrary invertible linear transformation)?

No. All we used is the existence of the inverse. (However, in the general case it would be nonstandard and probably confusing to denote the matrix and its inverse by the same letter without a "$^{-1}$".) The formula even applies to a nonlinear (curvilinear) coordinate transformation if you take

$$\Lambda^{\overline{\alpha}}{}_{\beta} = \frac{\partial x^{\overline{\alpha}}}{\partial x^{\beta}}$$

(the Jacobian matrix, which is the matrix of a *linear* transformation of vectors based at the point concerned).

(e) Is the trace of a $\binom{0}{2}$ tensor ($\sum_{\alpha=0}^{3} T_{\alpha\alpha}$) independent of coordinate system? Explain.
No. This time we get

$$\sum_{\overline{\mu}} T_{\overline{\mu}\overline{\mu}} = \sum_{\overline{\mu}} \Lambda^{\kappa}{}_{\overline{\mu}} \Lambda^{\lambda}{}_{\overline{\mu}} T_{\kappa\lambda}\,,$$

and there is no reason why $\sum_{\overline{\mu}} \Lambda^{\kappa}{}_{\overline{\mu}} \Lambda^{\lambda}{}_{\overline{\mu}}$ should equal $\delta^{\kappa\lambda}$ in general (even if Λ is Lorentz). Here is an explicit counterexample in dimension 2:

$$\Lambda = \begin{pmatrix} C & S \\ S & C \end{pmatrix}, \quad T = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}.$$

$$\Lambda T \Lambda^{\mathrm{t}} = \begin{pmatrix} C^2 a + S^2 d & SCa + SCd \\ SCa + SCd & S^2 a + C^2 d \end{pmatrix},$$

$$\operatorname{tr}(\Lambda T \Lambda^{\mathrm{t}}) = (C^2 + S^2)(a+d) \neq \operatorname{tr} T = a + d.$$

For comparison, if $C = \cosh\theta$, $S = \sinh\theta$, then Λ is a Lorentz transformation, and if T represents a $\binom{1}{1}$ tensor, we have

$$\operatorname{tr} \Lambda T \Lambda^{-1} = (C^2 - S^2)(a+d) = a + d,$$

as expected.

2. *(Essay – 10 pts.)* In the Brans–Stewart periodic universe, an observer in motion ages more slowly than an observer at rest (as determined by comparison of clocks at their reunion). Yet each is always in uniform motion relative to the other, at the same speed. Explain why this does not contradict the "principle of relativity".

3. *(50 pts.)* In some region around the origin in two-dimensional Minkowski space, introduce new coordinates (τ, σ) by

$$t = \tau + \tfrac{1}{2}\epsilon\sigma^2\,, \qquad x = \tfrac{1}{2}\epsilon\tau^2 + \sigma,$$

where ϵ is some small constant.

(a) Find the metric tensor, $\{g_{\mu\nu}\}$, (or, equivalently, the line element, ds^2) in the new coordinates.

$$\tilde{E}^t \equiv dt = d\tau + \epsilon\sigma\, d\sigma, \quad \tilde{E}^x \equiv dx = \epsilon\tau\, d\tau + d\sigma. \qquad (*)$$

Thus

$$\begin{aligned} dx^2 &= -(dt)^2 + (dx)^2 \\ &= -d\tau^2 - 2\epsilon\sigma\, d\tau\, d\sigma - \epsilon^2\sigma^2\, d\sigma^2 \\ &\quad + \epsilon^2\tau^2\, d\tau^2 + 2\epsilon\tau\, d\tau\, d\sigma + d\sigma^2 \\ &= -(1-\epsilon^2\tau^2)d\tau^2 + 2\epsilon(\tau-\sigma)d\tau\, d\sigma + (1-\epsilon^2\sigma^2)d\sigma^2. \end{aligned}$$

In other words,

$$\mathsf{g} = \begin{pmatrix} -(1-\epsilon^2\tau^2) & \epsilon(\tau-\sigma) \\ \epsilon(\tau-\sigma) & 1-\epsilon^2\sigma^2 \end{pmatrix}.$$

(b) Find the tangent vectors to the coordinate curves, $\vec{e}_\tau$ and $\vec{e}_\sigma$.

$$\begin{aligned}\vec{e}_\tau &= \frac{\partial t}{\partial \tau}\,\vec{e}_t + \frac{\partial x}{\partial \tau}\,\vec{e}_x = \vec{e}_t + \epsilon\tau\vec{e}_x \,, \\ \vec{e}_\sigma &= \frac{\partial t}{\partial \sigma}\,\vec{e}_t + \frac{\partial x}{\partial \sigma}\,\vec{e}_x = \epsilon\sigma\epsilon_t + \vec{e}_x \,.\end{aligned} \qquad (\dagger)$$

These can be written in matrix notation as

$$\vec{e}_\tau = \begin{pmatrix} 1 \\ \epsilon\tau \end{pmatrix}, \quad \vec{e}_\sigma = \begin{pmatrix} \epsilon\sigma \\ 1 \end{pmatrix}.$$

(c) Find the Christoffel symbols ($\Gamma^\tau_{\sigma\sigma}$, etc.).

The basic equation here is

$$\frac{d\vec{e}_\nu}{dx^\mu} = \Gamma^\lambda_{\mu\nu}\vec{e}_\lambda \,,$$

where $x^\mu \equiv (\tau, \sigma)$. We find

$$\Gamma^\lambda_{\tau\sigma}\vec{e}_\lambda = \frac{d\vec{e}_\tau}{d\sigma} = 0, \quad \Gamma^\lambda_{\sigma\tau}\vec{e}_\lambda = \frac{d\vec{e}_\sigma}{d\tau} = 0,$$

$$\Gamma^\lambda_{\tau\tau}\vec{e}_\lambda = \frac{d\vec{e}_\tau}{d\tau} = \epsilon\vec{e}_x \,, \quad \Gamma^\lambda_{\sigma\sigma}\vec{e}_\lambda = \frac{d\vec{e}_\sigma}{d\sigma} = \epsilon\vec{e}_t \,.$$

The first two equations (actually, either one of them) yield

$$\Gamma^\lambda_{\tau\sigma} = 0 = \Gamma^\lambda_{\sigma\tau} \quad \text{for all } \lambda \,.$$

For the other two equations we need to do more work. Solve the equations $(\dagger)$:

$$\vec{e}_t = (1-\epsilon^2\tau\sigma)^{-1}(\vec{e}_\tau - \epsilon\tau\vec{e}_\sigma), \quad \vec{e}_x = (1-\epsilon^2\tau\sigma)^{-1}(-\epsilon\sigma\vec{e}_\tau + \vec{e}_\sigma).$$

Substituting these into the remaining Christoffel equations and comparing with the basie equation, we see

$$\Gamma^\tau_{\tau\tau} = \frac{-\epsilon^2\sigma}{1-\epsilon^2\tau\sigma}\,, \quad \Gamma^\sigma_{\sigma\sigma} = \frac{-\epsilon^2\tau}{1-\epsilon^2\tau\sigma}\,,$$

$$\Gamma^\sigma_{\tau\tau} = \frac{\epsilon^2}{1-\epsilon^2\tau\sigma}\,, \quad \Gamma^\tau_{\sigma\sigma} = \frac{\epsilon^2}{1-\epsilon^2\tau\sigma}\,.$$

(d) Find the normal one-forms to the coordinate “surfaces”, $\tilde{d}\tau$ and $\tilde{d}\sigma$ (also called $\tilde{E}^\tau$ and $\tilde{E}^\sigma$).

We need to solve the equations $(*)$. The matrix to be inverted is the transpose of the one we inverted to solve $(\dagger)$, so we immediately get

$$\begin{aligned}\tilde{E}^\tau &= (1-\epsilon^2\tau\sigma)^{-1}(\tilde{E}^t - \epsilon\sigma\tilde{E}^x) = (1-\epsilon^2\tau\sigma)^{-1}(1, -\epsilon\sigma), \\ \tilde{E}^\sigma &= (1-\epsilon^2\tau\sigma)^{-1}(-\epsilon\tau\tilde{E}^t + \tilde{E}^x) = (1-\epsilon^2\tau\sigma)^{-1}(-\epsilon\tau, 1).\end{aligned}$$

(e) Verify that the basis you found in (d) is dual to the basis you found in (b). (Explain what "dual" means in this context.)

We want to check that $\tilde{E}^\lambda(\vec{e}_\kappa) = \delta^\lambda_\kappa$ in all cases. This is easily done with the row and column matrix forms we found:

$$\tilde{E}^\tau(\vec{e}_\tau) = (1-\epsilon^2\tau\sigma)^{-1}(1, -\epsilon\sigma)\begin{pmatrix} 1 \\ \epsilon\tau \end{pmatrix} = 1,$$

$$\tilde{E}^\tau(\vec{e}_\sigma) = (1-\epsilon^2\tau\sigma)^{-1}(1, -\epsilon\sigma)\begin{pmatrix} \epsilon\sigma \\ 1 \end{pmatrix} = 0,$$

and similarly for the other two.

(f) *(Bonus question – 5 points)* Can you say anything about how far from the origin we can go before something "goes wrong" with this coordinate system? *Suggestion:* Look at the determinant of the metric tensor.

$$\begin{aligned} \det g &= -(1-\epsilon^2\tau^2)(1-\epsilon^2\sigma^2) - \epsilon^2(\tau-\sigma)^2 = [\text{seven terms}] \\ &= -1 + 2\epsilon^2\tau\sigma - \epsilon^4\tau^2\sigma^2 = -(1-\epsilon^2\tau\sigma)^2. \end{aligned}$$

Therefore, the metric is singular wherever $\tau\sigma = \epsilon^{-2}$ (and nowhere else, since the metric components are never singular). This locus is a hyperbola in the (τ, σ) plane; the origin sits in a region between the two branches of the hyperbola, inside which the metric is regular. It would be nicer to have formulas for the images of these boundary curves in the (t, x) plane, but that seems hard to get. The region is large, because we assumed ϵ to be small.