Jordan Rogers

Homework 3 Revision Oct 28, 2009

p146 ① Given $A*B*C$ and $A*C*D$

ⓐ Prove that A, B, C, and D are four distinct points.

Proof: According to B-1, if we have $A*B*C$, then we can say that A, B, and C are distinct points all lying on the same line. Also, we have $A*C*D$, so A, C, and D are distinct points lying on the same line. Using RAA, assume $B=D$. B-3 states that only one point lies between the other two. But if B is between A and C, so is D. This is a contradiction to B-3. Similarly, for C to be between A and D, $B \neq D$ because B is between A and C. Therefore A, B, C, and D are four distinct points

ⓑ Prove that A, B, C, and D are collinear.

Proof: Betweeness axiom 1 states that if we have $A*B*C$, we can say that A, B, and C lie on the same line, l. Using the same axiom, we can say that A, C, and D lie on the same line, n, because we are given $A*C*D$. By incidence axiom 1, there is a unique line incident with 2 points (A and C). So l must equal n. Therefore, A, B, C, and D are collinear.

ⓒ Prove the corollary to Axiom B-4.

Proof: The corollary to Axiom B-4 says that if A and B are on opposite sides of l and if B and C are on the same side of l, then A and C are on opposite sides of l. A and B are on opposite sides of l, so by the definition of opposite sides, $\overline{AB}$ intersects l. By the definition of same side, since B and C are on the same side, the segment $\overline{BC}$ does not contain a point on the line l. Using RAA, let us assume that A and C lie on the same side of l. Then by axiom B-4 i, A, B, and C all lie on the same side of ℓ. This is a contradiction to our statement that A and B lie on opposite sides of ℓ. By RAA, A and C lie on opposite sides of line ℓ.

(2) (c) Prove the converse of Proposition 3.3 by applying Axiom B-1.

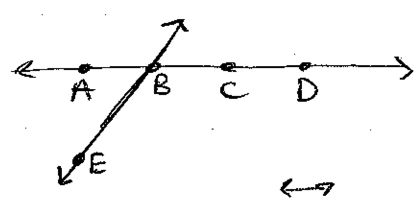
Proof: Proposition 3.3 states that if we are given $A*B*C$ and $A*C*D$, then we have $B*C*D$ and $A*B*D$.

The converse of this statement is: If we have $B*C*D$ and $A*B*D$, then we have $A*B*C$ and $A*C*D$.

By axiom B-1, if we have $B*C*D$, then we know that B, C, and D are distinct points that lie on the same line. Similarly, A, C, and D lie on the same line because we have $A*C*D$.

we know that C lies strictly between B and D by axiom B-3, and B lies strictly between A and D by the same axiom. so, by exercise 1, we know that A, B, C, and D are 4 distinct points.

By proposition 2.3, there exists a point E not on the line through A, B, C, and D. Let us consider the line $\overleftrightarrow{EB}$



By hypothesis, since $\overleftrightarrow{AD}$ intersects $\overleftrightarrow{EB}$ at point B, A and D are on opposite sides of $\overleftrightarrow{EB}$ by the definition of opposite sides. Also, we claim that C and D lie on the same side of $\overleftrightarrow{EB}$. Using RAA, let us say that C and D are on opposite sides of $\overleftrightarrow{EB}$. Then $\overleftrightarrow{CD}$ intersects $\overleftrightarrow{EB}$ at a point between C and D by definition of opposite sides. By proposition 2.1, that point must be B. Therefore, $C*B*D$, but this contradicts the given statement of $B * C * D$ and axiom B-3. Thus C and D are on the same side of $\overleftrightarrow{EB}$ by RAA.

So A and C are on opposite sides of $\overleftrightarrow{EB}$ by the corollary of axiom B-4. Therefore, point B is the intersection of lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{EB}$ and it lies between A and C by definition of opposite sides and proposition 2.1.

Thus, if $B * C * D$ and $A * B * D$, then $A * B * C$ and $A * C * D$.

(3) Given $A * B * C$

(a) Use proposition 3.3 to prove that $AB \subset AC$. Interchanging A and C, deduce $CB \subset CA$; which axiom justifies this interchange?

Let $X \in AB$. Then there are three cases to consider. First, $X = A$: By definition of a line segment, if $X = A$, then $X \in AC$. Second, $X = B$: Also by definition of a line segment, $X \in AC$ because $A * B * C$ and $B \in AC$. In the last case, $A * X * B$ and $X \neq A$ and $X \neq B$. Using proposition 3.3, if $A * X * B$ and $A * B * C$, then $X * B * C$ and $A * X * C$, so $X \in AC$ and $AB \subset AC$.

5/5

If we interchange A and C we have $CB \subset CA$. Axiom B-1 allows us to interchange A and C because $A * B * C$ is the same as $C * B * A$. So, using the same reasoning we used to deduce $AB \subset AC$, we can deduce $CB \subset CA$. ✓

(b) Use Axiom B-4 to prove that $AC \subset AB \cup BC$.

Let M be a point on segment AC and let there be a line l through P that is not the same line that goes through A, B, and C. So A, B, and C do not lie on l and there are two cases:

5/5

Case 1: A and B are on opposite sides of l and A and C are on opposite sides of l. Then, by axiom B-4, B and C are on the same side of l. Thus $P * B * C$ by proposition 3.3 (since $A * B * C$ and $A * P * C$) So we can conclude $A * P * B$ by proposition 3.3, and $P \in AB$.

Case 2: Suppose B and C are on opposite sides of l and A and C are on opposite sides of l. Then, by axiom B-4, A and B are on the same side of ℓ. So $A*B*P$ (since $A*B*C$ and $A*P*C$) Therefore we can conclude $B*P*C$ by proposition 3.3. So $P \in BC$.

Therefore, by considering both cases, $P \in AB \cup BC$, so $AC \subset AB \cup BC$.

④ Given $A*B*C$

ⓐ If P is a fourth point collinear with $A, B,$ and C, use proposition 3.3 and an axiom to prove that $\sim A*B*P \rightarrow \sim A*C*P$

By hypothesis, P is not between A and P. By axiom 2, either $P*A*B$ or $A*P*B$.

Case 1 : $P*A*B$

5/5 We can say $P*A*C$ by proposition 3.3 because we know $A*B*C$. So this is not $A*C*P$ and it holds true.

Case 2 : $A*P*B$

Knowing $A*B*C$, we can say $A*P*C$ by proposition 3.3. This is not $A*C*P$, so this holds true.

Either case gives us $\sim A*B*P \rightarrow \sim A*C*P$

ⓑ Deduce that $\overrightarrow{BA} \subset \overrightarrow{CA}$ and, symmetrically, $\overrightarrow{BC} \subset \overrightarrow{AC}$.

Let $M \in \overrightarrow{BA}$. By definition of a ray $\overrightarrow{BA}$, $M=B$, or $M=A$, or $M \in AB$ or $B*A*M$.

Case 1 : $M=B$

We are given $A*B*C$, and by axiom 1, this is the same as $C*B*A$. By definition of a ray $B \in \overrightarrow{CA}$.

5/5 So $M \in \overrightarrow{CA}$

Case 2 : $M=A$

By definition of a ray, $M \in \overrightarrow{CA}$

Case 3 : $M \in AB$ $(M \neq A, M \neq B)$

Given $A*B*C$, we have $C*B*A$ by axiom 1. Since $M \in AB$, then, by definition of a segment, $A*M*B$, which is the same as $B*M*A$ by axiom 1. By proposition 3.3, $C*M*A$, so $M \in CA$ and by definition of a ray $M \in \overrightarrow{CA}$

case4: $B * A * m$

We have $C * B * A$ by axiom 1 because we are given $A * B * C$. We know $C * B * A$ and $B * A * m$. Thus, by the corollary to proposition 3.3, $C * A * m$. So, by the definition of a ray, $m \in \overrightarrow{CA}$.

So, in any case, $m \in \overrightarrow{CA}$ and $\overrightarrow{BA} \subset \overrightarrow{CA}$. Therefore, by axiom B-1 and by interchanging A and C, and then using a similar 4 case argument, $\overrightarrow{BC} \subseteq \overrightarrow{AC}$.

(c) Use this result, proposition 3.1(a), proposition 3.3, and proposition 3.5 to prove that B is the only point that $\overrightarrow{BA}$ and $\overrightarrow{BC}$ have in common.

We are given $A * B * C$, and from part b, $\overrightarrow{BA} \subset \overrightarrow{CA}$ and $\overrightarrow{BC} \subset \overrightarrow{AC}$. So $\overrightarrow{CA} \cap \overrightarrow{AC} = AC$ from proposition 3.1(a). From proposition 3.5, $AC = AB \cup BC$ and B is the only point in common to segments AB and BC. So $AB \cap BC = \{B\}$. Now, we need to prove that B is the only point in common in $\overrightarrow{BA} \cap \overrightarrow{BC}$. So $\overrightarrow{BA} = BA \cup \{P : B * A * P\}$, thus $P \notin BC$ and $\overrightarrow{BC} = BC \cup \{P : B * C * P\}$ and $P \notin BA$ by the definition of a ray. Then $P * A * B$ and $B * C * P$ is a contradiction to proposition 3.3. So, P must be in the segment and is the only point in common with $\overrightarrow{BA} \cap \overrightarrow{BC}$.

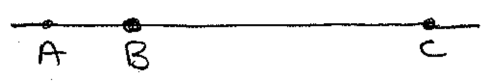
(15) Find an interpretation in which the incidence axioms and the first two betweeness axioms hold but Axiom B-3 fails in the following way: There exist three collinear points, no one of which is between the other two.

Proof: Let the betweeness relation, A*B*C, mean that B is the midpoint of AC. changing the definition of A*B*C does not affect the definitionsof the three incidence axioms because the new definition is Euclidean geometry and is not related to betweeness.

For Betweeness Axiom 1, A*B*C let us say that A, B, and C are distinct points on the same line and B is the midpoint of AC, so C*B*A.

For Betweeness Axiom 2, take two points, B and D. Then there exists a midpoint, C, of the segment BD. There is also another point, E, such that D is the midpoint of B and E and there exists a point, A, such that B is the midpoint of A and D. So axiom 2 still holds.

For Betweeness axiom 3, if we have points A, B, and C, we could have a situation like this:



Here, B is not the midpoint of $\overline{AC}$, so there is no betweeness relation among these 3 points according to our new definition.

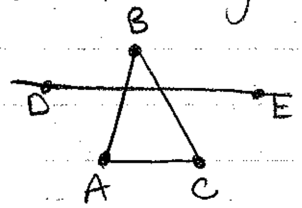
(18) Prove that a half-plane, the interior of an angle, and the interior of a triangle are all convex sets, where as the exterior of a triangle is not convex.

<u>Proof</u>: By definition of a half-plane, a half-plane is the set of all points on the same side of a line l as a point A not lying on l. So there exists a line l such that point A does not lie on l. Then let P be a point not on l such that P lies on the same side of l as A. This is the half-plane, H_A, of A. Let C be on $\overline{AP}$ such that $A * C * P$. Then by the definition of same sides, C does not lie on l and since $C \in \overline{AP}$, then C is also on the same side of l as A and P. So $C \in H_A$. Since A and P are in H_A and $A * C * P$ is in H_A, then H_A is convex.

Let there be an angle $\sphericalangle CAB$, and let D and E be distinct points in the interior of $\sphericalangle CAB$. Since $\sphericalangle CAB$ is the intersection of two half-planes, then the intersection of convex sets is convex, so the interior of an angle is a convex set.

Let there be three points, A, B, and C that create ΔABC and let points D and E lie in the interior of ΔABC. We know that the interior of ΔABC is the intersection of the interior of $\sphericalangle A$, $\sphericalangle B$, and $\sphericalangle C$, so the intersection of convex sets is convex. So D and E lie in a convex set.

To consider the exterior of a triangle, let us look at two distinct points, D and E, that lie on the exterior of ΔABC. Let D be on the exterior of AB and let E be on the exterior of BC. Now, let there be a unique line, ℓ, that passes through D and E.



In order for the exterior of a triangle to be convex, the entire segment DE must be contained within the set (the exterior). But part of $\overline{DE}$ lies on the interior of $\triangle ABC$, so the exterior of a triangle is not convex.