Jordan Rogers
Nov 16, 2009

Homework 4 Revison

(26) Justify each step in Proposition 3.17.

Proof:

① There is a unique point B' on ray $\overrightarrow{DE}$ such that $DB' \cong AB$ by congruence axiom 1.

② $\triangle ABC \cong \triangle DB'F$ by side-angle-side (congruence axiom 6)

③ Hence, $\sphericalangle DFB' \cong \sphericalangle C$ by definition of congruence

④ This implies $\overrightarrow{FE} = \overrightarrow{FB'}$ by congruence axiom 4

⑤ In that case, $B' = E$ because of step 4 and the point in common, F.

⑥ Hence, $\triangle ABC \cong \triangle DEF$ by congruence axiom 6 (SAS) and steps 2 and 5.

(27) Prove Proposition 3.18.

Proposition 3.18 states that if in $\triangle ABC$ we have $\sphericalangle B \cong \sphericalangle C$, then $AB \cong AC$ and $\triangle ABC$ is isosceles.

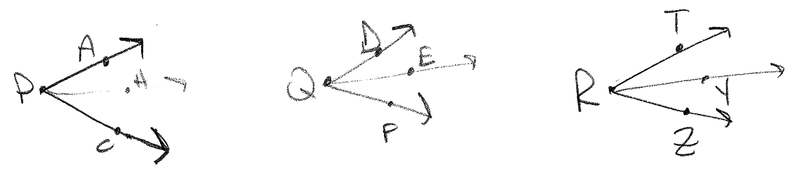
Proof:

① Consider the correspondence of vertices $A \leftrightarrow A$, $B \leftrightarrow C$, $C \leftrightarrow B$. Under this correspondence, the two angles $\sphericalangle ABC$ and $\sphericalangle ACB$ and the included side of $\triangle ABC$ are congruent, respectively, to the corresponding angles and included side of $\triangle ACB$ (by hypothesis and congruence axiom 5)

② Hence, $\triangle ABC \cong \triangle ACB$ (by ASA) so $AB \cong AC$ (by definition of congruence of triangles)

③ Thus, $\triangle ABC$ is isosceles (by definition of isosceles)

Figures for 31:



(31) Prove Proposition 3.21.

A) Exactly one of the following three conditions holds: $\measuredangle P < \measuredangle Q$, $\measuredangle P \cong \measuredangle Q$, or $\measuredangle Q < \measuredangle P$.

B) If $\measuredangle P < \measuredangle Q$ and $\measuredangle Q \cong \measuredangle R$, then $\measuredangle P < \measuredangle R$.

C) If $\measuredangle P > \measuredangle Q$ and $\measuredangle Q \cong \measuredangle R$, then $\measuredangle P > \measuredangle R$.

D) If $\measuredangle P < \measuredangle Q$ and $\measuredangle Q < \measuredangle R$, then $\measuredangle P < \measuredangle R$.

Proof:

A) Let $\measuredangle APC$ and $\measuredangle DQF$ be any two angles. Given $\measuredangle APC$ and any ray $\overrightarrow{QD}$ emanating from Q, there exists a unique ray $\overrightarrow{QE}$ on a given side of $\overrightarrow{QD}$ such that $\measuredangle APC \cong \measuredangle DQE$ (by C-4). Then, there are 3 positions for $\overrightarrow{QE}$:

Case 1: $\overrightarrow{QE} \cong \overrightarrow{QF}$

Then $\measuredangle APC \cong \measuredangle DQE \cong \measuredangle DQF$, so $\measuredangle P \cong \measuredangle Q$.

Case 2: $\overrightarrow{QE}$ lies on the interior of $\measuredangle DQF$ so $\overrightarrow{QE}$ is between $\overrightarrow{QF}$ and $\overrightarrow{QD}$, and $\measuredangle APC \cong \measuredangle DQE$, then $\measuredangle APC < \measuredangle DQF$ (by definition of less than).

Case 3: $\overrightarrow{QE}$ lies on the exterior of $\measuredangle DQF$ so $\overrightarrow{QF}$ is between $\overrightarrow{QE}$ and $\overrightarrow{QD}$, and $\measuredangle APC \cong \measuredangle DQE$, then $\measuredangle DQF < \measuredangle DQE$ (by definition of less than).

Thus, by transitivity, $\measuredangle DQF < \measuredangle APC$

So, exactly one case holds: either $\measuredangle P < \measuredangle Q$, $\measuredangle P \cong \measuredangle Q$, or $\measuredangle Q < \measuredangle P$.

B) If $\measuredangle P < \measuredangle Q$ and $\measuredangle Q \cong \measuredangle R$, then $\measuredangle P < \measuredangle R$:

Let $\measuredangle APC$ and $\measuredangle DQF$ be any two angles such that $\measuredangle APC < \measuredangle DQF$. Then there exists a ray $\overrightarrow{QE}$ on the interior of $\measuredangle DQF$ such that $\measuredangle APC \cong \measuredangle EQF$. Let $\measuredangle XRZ$ be an angle such that $\measuredangle XRZ \cong \measuredangle DQF$. Since $\measuredangle EQF < \measuredangle DQF$ (by def. of less than), then $\measuredangle EQF < \measuredangle XRZ$ (since $\measuredangle Q \cong \measuredangle R$). Since $\measuredangle APC \cong \measuredangle EQF$, then $\measuredangle APC < \measuredangle XRZ$.

C) If $\measuredangle P > \measuredangle Q$ and $\measuredangle Q \cong \measuredangle R$, then $\measuredangle P > \measuredangle R$:
Let $\measuredangle APC$ and $\measuredangle DQF$ be any two angles such that $\measuredangle DQF < \measuredangle APC$. Then there exists a ray $\overrightarrow{PH}$ on the interior of $\measuredangle APC$ such that $\measuredangle DQF \cong \measuredangle HPC$ (by def. of less than). Let $\measuredangle XRZ$ be any angle such that $\measuredangle XRZ \cong \measuredangle DQF$. Then $\measuredangle XRZ \cong \measuredangle DQF \cong HPC$. Therefore, $\measuredangle XRZ < \measuredangle APC$.

D) If $\measuredangle P < \measuredangle Q$ and $\measuredangle Q < \measuredangle R$, then $\measuredangle P < \measuredangle R$.
Let $\measuredangle APC$, $\measuredangle DQF$, and $\measuredangle XRZ$ be any three angles such that $\measuredangle APC < \measuredangle DQF$ and $\measuredangle DQF < \measuredangle XRZ$. There exists a ray $\overrightarrow{QE}$ on the interior of $\measuredangle DQF$ such that $\measuredangle EQF \cong \measuredangle APC$ (by def. of less than).
?
There also exists a ray $\overrightarrow{RY}$ on the interior of $\measuredangle XRZ$ such that $\measuredangle YRZ \cong \measuredangle EQF$. Then, by transitivity, $\measuredangle APC \cong \measuredangle YRZ$. Finally, $\measuredangle APC < \measuredangle XRZ$ (by def. of less than).

(32) Prove Proposition 3.22

Proposition 3.22 states that if $AB \cong DE$, $BC \cong EF$, and $AC \cong DF$, then $\triangle ABC \cong \triangle DEF$.

Proof:

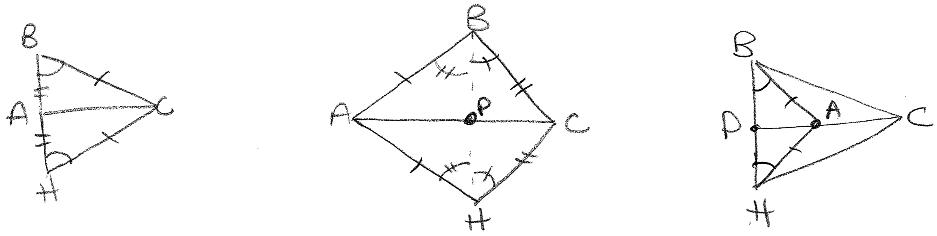
① By CA-4 and the corollary to SAS, there exists a unique point H on a given side of $\overleftrightarrow{AC}$ such that $\triangle DEF \cong \triangle AHC$.

② Since $BC \cong HC$, then $\triangle BCH$ is isosceles (by def. of isosceles triangles)

③ BH intersects line AC at a point P (by B and H being on opposite sides of $\overleftrightarrow{AC}$)

④ Then, there are 3 possible positions of P:
① $P = A$. Since $BC \cong HC$, $AB \cong AH$, and $\measuredangle CBA \cong \measuredangle CHA$ (by the fact that BCH is isosceles), then $\triangle ABC \cong \triangle AHC$. Thus, $\triangle ABC \cong \triangle DEF$ (by SAS and opposite sides)

Figures for 32



② $A * P * C$. Since $AB \cong AH$, $BC \cong HC$, $\triangle BCH$ and $\triangle ABH$ are isosceles. So, $\sphericalangle CBP \cong \sphericalangle CHP$ and $\sphericalangle ABP \cong \sphericalangle AHP$ (by def. isosceles triangle). Then $\sphericalangle ABC \cong \sphericalangle AHC$ (prop 3.19). Thus, $\triangle ABC \cong \triangle AHC$ and $\triangle ABC \cong \triangle DEF$ (by step 1 & SAS)

③ $P * A * C$. Since $AB \cong AH$, $\triangle BAH$ is isosceles and $\sphericalangle ABP \cong \sphericalangle AHP$ (by def of isosceles triangle.) Also $\sphericalangle CBP \cong \sphericalangle CHP$ (since $\triangle BCH$ is isosceles). Then, $\sphericalangle CBA \cong \sphericalangle CHA$ (by prop. 3.20), $\triangle ABC \cong \triangle AHC$, and $\triangle ABC \cong \triangle DEF$ (step 1 and SAS)

In all three cases, $\triangle ABC \cong \triangle DEF$

34 Ⓐ Euclid's Second Postulate: For every segment AB for every segment CD, there exists a unique point E on line $\overleftrightarrow{AB}$ such that B is between A and E and segment CD is congruent to BE.

Proof:

① Since C and D are distinct points, then for the ray opposite $\overrightarrow{BA}$, there exists a unique point E on that ray such that $CD \cong BE$ (C-1).

② Suppose B is not between A and E (RAA hypothesis)

③ Then either $A * E * B$ or $E * A * B$. In both cases, E lies on $\overrightarrow{BA}$, and so does not lie on the ray opposite $\overrightarrow{BA}$ (by def of ray and opposite rays).

④ Thus, $A * B * E$ (by 3, which contradicts step 2)

Ⓑ center:

① Let O and O' both be the centers of the same circle (RAA hypothesis)

② The line $\overleftrightarrow{OO'}$ intersects the circle at two points (by line-circle continuity). Let's call these points A and B. Let $A * O * O' * B$.

③ $AO \cong OB$ and $AO' = O'B$ (by definitions of center of a circle and radius.

④ A, O, O', and B are collinear (by 2 and B-1), and $AO < AO'$ and $O'B < OB$ (by def of < for segments)

⑤ Since $AO < AO'$ and $AO' \cong O'B$, then $AO < O'B$ (by prop 3.13)

⑥ Since $O'B < OB$ and $OB \cong AO$, then $O'B < AO$ (by prop. 3.13)

⑦ Thus, there is only one unique center in a circle (since 5 contradicts step 6)

Radius:

① Given center O and point A lying on the circle, $OA \cong OP$ for any other distinct point P on the circle (by def. of radius)

② Let center O' and point A lying on the circle determine the same circle. Since the center of a circle is unique, $O = O'$ and $OA \cong O'A$.

③ Thus, the radius of a circle is unique.