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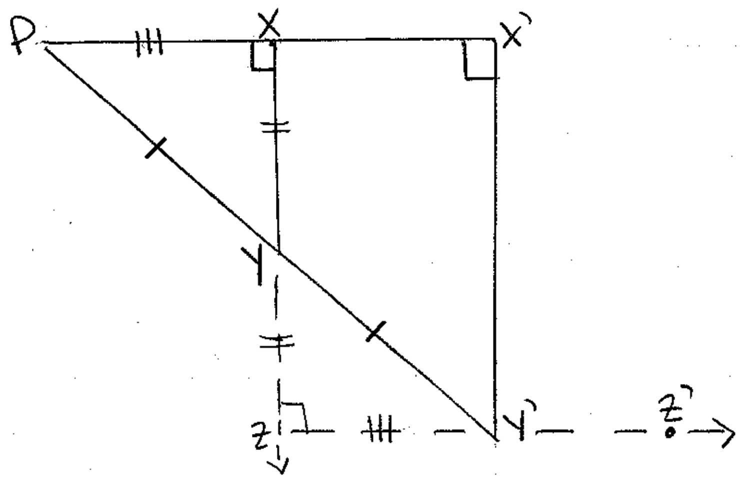
# Homework 6 Revision

(5.1)




Proof:

Part 1

① Given $\triangle PXY$ with right angle at X

② There exists a ray $\overrightarrow{XZ}$ emanating from X such that $X * Y * Z$ by B-2 and the definition of a ray.

③ construct Z such that $XY \cong YZ$ (by C-1)

④ Construct line $\overleftrightarrow{ZZ'}$ perpendicular to $\overrightarrow{XZ}$ at point Z (prop 3.16)

⑤ There exists Y' on line $\overleftrightarrow{ZZ'}$ such that $Z * Y' * Z'$ and also such that $ZY' \cong PX$ by definition of betweeneness and C-1

⑥ Thus, by SAS, $\triangle PXY \cong \triangle Y'ZY$

⑦ Now we know that $P * Y * Y'$ because Y lies on $\overrightarrow{XZ}$ which is perpendicular to $\overrightarrow{ZZ'}$ which Y' lies on.

⑧ By definition of congruent triangles, $PY' = 2PY$

Part 2

① Now we have a lambert quadrilateral with right angles at $\measuredangle PXY$, $\measuredangle PX'Y'$, and $\measuredangle XZY'$ (by definition of lambert quadrilateral, hypothesis and step 4 from Part 1)

② By Prop. 4.13 corollary 3, if $\measuredangle X'Y'Z$ is acute then $X'Y' > XZ$

③ Thus by Part 1, we know $XZ \cong 2XY$ and by substitution, $X'Y' \geq 2XY$

④ Therefore, $X'Y'$ is the side opposite $\measuredangle P$ and is at least double $XY$

⑤ By Corollary 3 of Prop 4.13, if $\measuredangle X'Y'Z$ is a right angle, then $Y'Z \cong XX'$ and if $\measuredangle X'Y'Z$ is acute, then $Y'Z > XX'$

⑥ By step 5, $Y'Z \geq XX'$

⑦ Now by part 1 we know $Y'Z \cong PX$, so by substitution $PX \geq X'X$

⑧ Therefore, $PX' = PX + XX'$ by addition of segments

⑨ Since $PX \geq XX'$, then $PX' \leq 2PX$ (by step 7)

⑩ Therefore $PX'$ is at most double $PX$ by step 9

5.7 Gergonne proves that there is no last ray between $\overrightarrow{PA}$ and $\overrightarrow{PQ}$ that intersects $l$. However, he has a flaw because this does not necessarily imply his conclusion that all rays between $\overrightarrow{PA}$ and $\overrightarrow{PQ}$ meet $l$ There could be a line that does not even meet $l$, like a parabolic line. this proving that there is no last ray does not prove his conclusion. He should have also taken into consideration the first ray that does not intersect $l$. He could also consider if there is no last ray that intersects or no first ray that does, but this fails because of Dedikind's model.

(6.6) Justify the steps that have not been justified.

(1) By definition of limiting parallel rays
(2) By definition of betweeness of rays and by definition of the interior or an angle.
(3) By B-3 and hypothesis $P * X * Y$ and $\overleftrightarrow{XR}$ intersects $\overrightarrow{PY}$ at X, then by definition of opposite sides, P and Y are on opposite sides
(4) By exercise 5
(5) By definition of betweeness of rays and definition of interior rays, $\overrightarrow{XS}$ is between $\overrightarrow{XR}$ and $\overrightarrow{XY}$, thus R and S are on the same side of $\overrightarrow{XY}$ and since X and Y are collinear, then $\overrightarrow{XY} = \overrightarrow{PY}$
(6) By definition of limiting parallel rays and that Q and R lie on $l$.
(7) By steps 5 and 6, and B-4
(8) By definition of betweeness of rays and definition of the interior of angle $\measuredangle YPQ$ and steps 4 and 7; then also by definition that PY is a limiting parallel ray.
(9) By step 8, S lies between P and T; by step 7, Q and S are on the same side of $\overleftrightarrow{PY}$; by hypothesis X lies on $\overrightarrow{PY}$ such that $\overleftrightarrow{PX} = \overleftrightarrow{XY}$; by definition of exterior of a triangle, X is the exterior of $\triangle PQT$
(10) All points on PQ are on the same side of XR because PQ and XR are both perpendicular to $l$. Since P and Y are on opposite sides by step 3, then all points on PQ are on the opposite side of XR as Y, and since S and Y are on the same side of XR by steps 1 and 2, and B-4
(11) By proposition 3.9a, $\overrightarrow{XS}$ either intersects PQ or QT. Thus, by step 10, $\overrightarrow{XS}$ must intersect QT.

(6a) ① Let $\overrightarrow{PZ}$ be between $\overrightarrow{PQ}$ and $\overrightarrow{PX}$, then $Z$ is in the interior of $\measuredangle QPX$ (by definition of interior angle and betweeness of rays)

② Choose any $S$ such that $S * P * Z$ (B-2)

③ we know $S$ and $Z$ are on opposite sides of line $\overleftrightarrow{PQ}$ by step 2 and by definition of opposite sides.

④ $Z$ and $X$ are on same sides of $\overleftrightarrow{PQ}$ by step 1 and definition of interior of an angle.

⑤ Then $S$ and $X$ are on opposite sides of $\overleftrightarrow{PQ}$ by steps 3, 4, and B-4

⑥ Thus, $SX$ meets $\overleftrightarrow{PQ}$ at a point $u$ by step 5

⑦ $u * P * Q$ by step 2, $S * P * Z$ and step 1. Since $Z$ is in the interior of $\measuredangle QPX$, so $S$ is in the exterior of $\measuredangle QPX$, thus $u$ is in the exterior of $\measuredangle QPX$.

⑧ Let $\overleftrightarrow{XR}$ be perpendicular to $l$ with $R$ intersecting $l$. Also, choose any $W$ such that $u * X * W$ by B-2

⑨ Then $u$ and $w$ are on opposite sides of $\overleftrightarrow{XR}$ by step 8.

⑩ By hypothesis, $P * X * Y$, so $P$ and $Y$ are on opposite sides of $\overleftrightarrow{XR}$ by definition of opposite sides.

⑪ Thus $W$ and $Y$ are on the same side of $\overleftrightarrow{XR}$ by steps 7, 9, and 10, and by definition of same side

⑫ $Q$ and $u$ are on opposite sides of $\overleftrightarrow{PY} = \overleftrightarrow{XY}$ by step 7 and the hypothesis that $P * X * Y$

⑬ $u$ and $W$ are on opposite sides of $\overleftrightarrow{PY} = \overleftrightarrow{XY}$ by step 8.

⑭ $Q$ and $W$ are on the same side of $\overleftrightarrow{PY}$ by steps 12 and 13 and by B-4.

(15) W and R are on the same side of $\overleftrightarrow{XY}$ because Q and R lie on $\ell$ and by the definition that $\overleftrightarrow{XY}$ is a limiting parallel ray to $\ell$.

(16) Thus W is in the interior of $\measuredangle YXR$ by steps 11 and 15 and the definition of the interior of an angle.

(17) $\overrightarrow{XW}$ is between $\overleftrightarrow{XY}$ and $\overleftrightarrow{XR}$ by Step16 and betweeness of rays.

(18) Thus $\overrightarrow{XW}$ meets $\ell$ at a point T by step 17 and the definition of limiting parallel rays.

(19) $\overrightarrow{PZ}$ meets $\ell$ in a point V by Prop. 3.9a

(20) $\overrightarrow{PY}$ is the limiting parallel ray to $\ell$ by step 19 and the definition of limiting parallel rays.

(6.10) Part 1

① Given $\triangle ABC$ and perpendicular bisector $l$ to $AB$ and perpendicular bisector $m$ to $BC$

② Assume $l \parallel m$ (RAA)

③ Then by prop 4.10, since $l \parallel m$ and $l \perp AB$ and $m \perp AB$, then either $AB = BC$ or $AB \parallel BC$.

④ By definition of a triangle, we know that $AB, BC, AC$ are 3 distinct non-collinear sides. So $BC \neq AB$

⑤ Since $AB$ and $BC$ have a point in common, then they are not parallel (by definition of parallel).

⑥ So we have a contraction and $l$ and $m$ meet at some point $D$. (RAA conclusion and by definition of parallel)

5/5 Part 2

① There exists a point $N$ on $AC$ such that $N$ is the midpoint on $AC$ (by definition of midpoint)

② Then there exists a ray $\overrightarrow{DN}$ by definition of a ray

③ Connect the vertices to the common point $D$ such that we create $BD, AC, DC$.

④ By SAS, we know $\triangle BMD \cong \triangle MCD$

⑤ By definition of congruent triangles, $BD \cong CD$

⑥ By SAS, $\triangle BDL \cong \triangle ADL$

⑦ By definition of congruent triangles, $BD \cong AD$

⑧ By steps 4 and 6, $AD \cong CD$

⑨ Thus by steps 5, 7, 8, $D$ is equidistant from the three vertices

⑩ By SSS, $\triangle ADN \cong \triangle CDN$

⑪ By definition of congruent triangles, $\angle CND \cong \angle AND$

(12) By step 11 and by definition of supplementary angles, $\measuredangle CND$ and $\measuredangle AND$ must be right angles

(13) By step 12 and since $AN \cong NC$, $\overrightarrow{DN}$ is perpendicular to $AC$

⑥ From the proof we know that $\triangle ABC$ has a defect $d$. Then since $\triangle DBC \cong \triangle ABC$, $\triangle DBC$ also has the same defect $d$. We also know that $\triangle AB_1C_1$ is composed of $\triangle ABC$, $\triangle DBC$, $\triangle BB_1D$, and $\triangle CC_1D$. Using Prop 6.1, $\triangle AB_1C_1$ is the sum of all these triangles. So we know that $\triangle ABC + \triangle DBC = 2d$, but we do not know if $\triangle BB_1D$ and $\triangle CC_1D$ have defects or not. Thus the sum of all the triangles

$\triangle ABC + \triangle DBC + \triangle BB_1D + \triangle CC_1D \geq 2d$

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