Math. 467 (Fulling) 15 December 2014

Final Examination – Solutions

Name: ______________________________

1. *(24 pts.)*

   (a) State the *alternate interior angle theorm.*

If the Hilbert IBC axioms hold, then *whenever two lines have a pair of congruent alternate interior angles with respect to a transversal line, those two lines are parallel.*

   (b) This theorem implies that parallel lines exist, so it is inconsistent with spherical or elliptic geometry. Sketch (at least) the proof of the theorem and identify which Hilbert axioms are used in the proof that are not valid in spherical or elliptic geometry.

[For the proof see p. 163.] The proof uses Axiom I-1 (in the form of its corollary, Prop. 2.1), which is false in spherical geometry, and Axiom B-4, which is false in elliptic geometry.

2. *(30 pts.)* Rearrange these names into historical order, earliest to latest:

Saccheri, Euclid, Thales, Gauss, Proclus, Klein

Thales, Euclid, Proclus, Saccheri, Gauss, Klein

3. *(18 pts.)* Critique each of these statements. (Is it true or false? Why?)

   (a) If the space we live in has constant positive curvature, then it is finite in extent, but if it has constant negative curvature, then it is infinite. (You may assume that what is true of 2-dimensional space in this regard is true of 3-dimensional space.)

The first half is true, but the second half is not. Just a torus has constant zero curvature but finite extent (by any reasonable definition of "extent"), there exist hyperbolic spaces with certain edges identified that have finite extent. Examples were shown in the paper of Levin and the presentation of Borthwick.

   (b) If you succeed in proving the parallel postulate, you will have shown that Euclidean geometry is *inconsistent.*

[Surely it's unnecessary for me to belabor this, since it's what the course was about. See pp. 289–293 or the corresponding pages of my notes on Chapter 7.]

4. *(32 pts.)*

   (a) Draw a picture illustrating why the Klein disk model satisfies the hyperbolic parallelism property (the negation of axiom HE).

   (b) Show, by pictures or words or both, that the Klein disk satisfies all the Hilbert I and B axioms. (There are 7 of them.)

[See Question 4 of the Fall 2013 final exam.]

5. *(16 pts.)* Prove **ONE** of these. (50% extra credit for the other one.)

   (a) The summit angles of a Saccheri quadrilateral are congruent to each other, and the line joining the midpoints of the summit and the base is perpendicular to both the summit and the base.

[See Prop. 4.12, pp. 177–178.]

(b) If $\square$ABDC is bi-right ($\angle$A $=$ $\angle$B $= 90^\circ$), then the greater side is opposite the greater angle ($\angle$C $>$ $\angle$D $\iff$ BD $>$ AC).

[See Prop. 4.13, pp. 178–179.]

6. *(Essay – 30 pts.)* Define *equivalence relation* and *partition*. Show how every equivalence relation determines a partion and vice versa. Discuss at least two examples of equivalence relations (and the corresponding partitions) that arose in this course.