Chapters 5 and 6

I won't repeat most of the history. That doesn't mean it's unimportant, only that I don't have anything to add. I'll review alternatives to the parallel postulate later.

Let's start with a remark on pp. 248–249: "A model of hyperbolic plane geometry is a sphere of imaginary radius with antipodal points identified in the three-dimensional spacetime of special relativity." This is better explained on pp. 311–313. I will come back to it later.

Results in non-Euclidean Hilbert geometry (pp. 250ff)

Negation of HE: $\exists$ a line l and a P not on l such that at least two distinct lines through P are $\parallel$ to l.

Def.: A Hilbert plane satisfying this is *non-Euclidean*.

Proclus's Theorem: If a Hilbert plane is semi-Euclidean (rectangles exist; triangle sums $= 180°$) and Aristotle's axiom holds, then HE holds. The converse is also true: Both parts were proved in Chap. 4.

Basic Theorem 6.1: A non-Euclidean Hilbert plane satisfying Aristotle's axiom satisfies the acute angle hypothesis (and hence a list of things we proved in the last chapter).

Remark: Recall that "rectangles do not exist" is a quick way of saying that either the acute angle hypothesis or the obtuse angle hypothesis holds. (See p. 182.)

Universal Non-Euclidean Theorem and Corollary: In a Hilbert plane in which rectangles do not exist, for every l and every P not on l there are at least 2 parallels to l through P (and hence infinitely many — cf. proof of Theorem 4.4).

Def.: The *defect* of a triangle is the amount by which the three angles fail to total $180°$.

Prop. 6.1: If a triangle is divided into 2 triangles, its defect is the sum of the defects of the parts.

Remark: In Chap. 10 it is proved that the *area* of a triangle is proportional to its defect. (The constant of proportionality is fixed by the scale factor of the hyperbolic geometry — the "radius" R of the hyperboloid in the model mentioned above.) It follows that in any particular hyperbolic geometry there is an upper limit on the area of triangles, since the defect can't be more than $180°$. The next proposition shows how this can be true, even though segments can be arbitrarily long:

Prop. 6.2: Acute hyp. $\Rightarrow$ if two triangles are similar, then they are congruent. (AAA is true!) [sketch] [The definition of "similar" is on pp. 215–216. For some reason this information is not in the book's index.]

Related remark: HE $\iff$ **Wallis's Axiom:** Given any segment and any triangle, there is a similar triangle built on that segment.

In effect, Wallis's axiom says that space looks the same at *all scales*. This is true of a Euclidean plane, but false of a sphere or a hyperboloid, where the radius sets the scale of curvature.

Prop. 6.3: In a plane in which rectangles do not exist, parallel lines are not equidistant. In fact, if $l' \| l$, then any set of points on l' equidistant from l contains at most two points. [I have reversed the labeling from the book's for later convenience.]

Related remark: semi-Euclidean $\iff$ **Clavius's Axiom:** The parallel through P is the equidistant locus through P (both relative to a line l).

Remark: "I hope the reader is not too shocked to see line l drawn as being 'curved'!" This should not be a shock to those who have drawn meridians and parallels on a sphere. [2 sketches] Great circles of constant longitude are not equidistant. Parallels of constant latitude are not great circles, except for the equator.

Limiting parallel rays

See Fig. 6.10 (p. 258), and Fig. 6.14 (p. 264), and Figs. 7.9–10 (p. 304); also Gergonne's flawed argument, Ex. 5.7, p. 231. Given l and P, some lines through P (those closest to the perpendicular) intersect l, others do not. Is there a boundary case between those that do and those that don't? Under sufficient continuity axioms, it can be proved that a boundary ray does exist and it does not intersect. In other words, there is a first ray that fails to meet l. Cf. least upper bound vs. maximum, and the possible nonexistence of either if $\mathbf{R}$ is replaced by $\mathbf{Q}$. (Greenberg's theorem uses Aristotle and line-circle; Dedekind would do for both.)

Hilbert made this property another axiom (p. 259):

Hilbert's Hyperbolic Axiom of Parallels: $\forall\, l$, P, a limiting parallel ray exists, and it is not $\perp$ to the $\perp$ from P to l.

Contrast the negation of HE, p. 250.

Definitions: A Hilbert plane obeying this axiom is a *hyperbolic plane*. A non-Euclidean plane satisfying Dedekind's axiom is a *real hyperbolic plane*.

Every plane satisfying the Hilbert IBC axioms either satisfies HE or does not, so all planes lie on one of the rows below (where the first row comes from the definitions above, and the second row from definitions in Chapter 3):

real hyperbolic	$\Rightarrow$	hyperbolic	$\Rightarrow$	non-Euclidean (Hilbert with $\neg$HE)
real Euclidean	$\Rightarrow$	Euclidean	$\Rightarrow$	Pythagorean (Hilbert with HE)

("Euclidean" was defined to include the assumption of circle-circle continuity, which is equivalent to line-circle continuity.*) The first two columns of this table describe more restrictive systems, characterized by continuity axioms:

Corollaries:

1. A Hilbert plane satisfying Dedekind's axiom is either real Euclidean or real hyperbolic.

2. A Hilbert plane satisfying Aristotle's axiom and line-circle continuity is either Euclidean or hyperbolic.

Theorem 6.3: In a hyperbolic plane, if m is $\parallel$ to l, then *either* m contains a limiting parallel ray in one direction or the other, *or* there is a (unique) common $\perp$ to l and m (but not *both*). (See Fig. 6.14, p. 264.)

Prop. 6.6: In a hyperbolic plane, given line l and point P (not on l), the angles between the perpendicular from P to l and the limiting rays from P to l are acute and are the same on both sides. (Their measure is called the *angle of parallelism* associated with that $\perp$ segment. Not surprisingly, it is determined by the length of the segment — see p. 332.)

Historical alternatives to the parallel postulate

1. Euclid ($\sim$300 BCE): Interior angles $< 180° \Rightarrow$ intersection (EV).

2. Proclus ($\sim$450), Playfair (1795): Parallels are unique (HE).

3. Clavius (1574): Parallel = equidistant.

* Greenberg's book shows only that circle-circle implies line-circle (Major Ex. 4.1), but Greenberg's recent article, M. J. Greenberg, Old and new results in the foundations of elementary plane Euclidean and non-Euclidean geometries, *Amer. Math. Monthly* **117** (2010) 198–219 (see p. 202), indicates that the converse has been known for some time but requires a more sophisticated proof.

4. Wallis (1693): Scale invariance (similar triangles always exist).

5. Clairaut (1741): Rectangles exist.

6. Legendre ($\sim$1800): $\forall$ acute angle A and $\forall$ D interior to angle A, $\exists$ a line through D and not through the vertex A that intersects both sides of angle A. (That is, the Warning on p. 115 should be defied.)

Most of these people thought they were *proving* EV because their substitute axioms were "obvious". They were wrong.

Dehn's models

[Table from Dehn's 1900 paper (table translated by G. B. Halsted, "Supplementary report on non-Euclidean geometry," *Science* **14** (1901) 705–717.)]

This situation makes necessary the complicated statement of hypotheses in the theorems about Saccheri–Lambert quadrilaterals.

See pp. 188-189 and 250, especially footnote on p. 189. My attempt to elaborate: (These remarks should be considered intuitive only.)

Start by noting that $\mathbf{F}^2$ and $\mathbf{F}^3$ model Euclidean 2D and 3D geometry, if $\mathbf{F}$ is an ordered field in which you can take square roots (Theorem, p. 141).

There are such fields containing "infinitesimal" elements, say a, such that

$$na \equiv \sum_{j=1}^{n} a < r \quad \text{for any } r \in \mathbf{Q}^+.$$

(Thus Archimedes's axiom fails for segments of such lengths.) The modern theory of *nonstandard analysis* attempts (among other things) to vindicate pre-19th-century calculus in this way (i.e., making equations like $dy = \frac{dy}{dx}\,dx$ literally true); it developed in the mid-20th century, hence 50 years after Dehn, but Dehn already knew about non-Archimedean fields. The apparent judgment of history is that calculus doesn't need this: the rigorous theory of limits is better. $\mathbf{Q}$ and $\mathbf{K}$ are too small; fields $\mathbf{F}$ with infinitesimals are two big; $\mathbf{R}$ is just right.

Nevertheless, a rigorous theory of infinitesimals allows one to create models of non-Legendrean and of semi-Euclidean (but not Euclidean) geometry. Let Π be the subset of $\mathbf{F}^2$ of points whose coordinates are infinitesimal. Since you can't get out of Π by adding elements of Π, the Hilbert axioms are still satisfied, so Π is also a model for them. Also, it satisfies the right angle hypothesis because $\mathbf{F}^2$ does. However, two lines in $\mathbf{F}^2$ determined by pairs of points in Π that intersect in a point not in Π count as parallel in Π. Therefore, there are lots of parallels; HE/EV fails although the plane is semi-Euclidean (rectangles exist; triangles add to 180°).

Next, consider a sphere in $\mathbf{F}^3$. (This is not precisely Dehn's construction, but seems to be consistent with Greenberg's footnote.) Consider an infinitesimal neighborhood of a point on the sphere. For example, introduce polar coordinates (θ, ϕ) around the north pole and restrict θ to infinitesimal values (getting a very tiny polar cap). Now most pairs of great circles do not intersect inside the cap, so they are parallel. However, the fourth angles of Lambert quads. inside the cap are obtuse (although only *infinitesimally* larger than a right angle).

References:

1. H. J. Keisler, *Elementary Calculus: An Infinitesimal Approach*, 2nd ed., Prindle, Weber & Schmidt (1986), `http://www.math.wisc.edu/~keisler/ calc.html`. (See the last part of Chapter 1.) This is a calculus textbook based on nonstandard analysis, so presumably it contains the most elementary exposition of nonstandard analysis that is available.

2. P. Ungar, [review of three calculus textbooks], *Amer. Math. Monthly* **93** (1986) 221–230. (See pp. 224–226.) This contains a scathing indictment of the nonstandard-analysis approach to *teaching* calculus — and of many other textbook sins as well.

3. K. Hrbacek, O. Lessmann, R. O'Donovan, Analysis with ultrasmall numbers, *Amer. Math. Monthly* **117** (2010) 801–816. (This may be regarded as an attempt to accommodate Ungar's criticism that "traditional" nonstandard analysis does not properly model how scientists and applied mathematicians use differentials. Whether it succeeds is not for me to say. Unfortunately, I don't think it is possible to construct Dehn's models within this new approach.)