Math. 467, Sec. 900 (Fulling) 12 December 2016

# Final Examination – Solutiona

Name: ______________________________

1. *(Multiple choice – each 5 pts.)* Circle the correct capital letter.

   (a) The ratio of a circle's circumference to its radius is
   - (A) always equal to $2\pi$.
   - (B) greater than $2\pi$ in elliptic geometry and less than $2\pi$ in hyperbolic geometry.
   - (C) less than $2\pi$ in elliptic geometry and greater than $2\pi$ in hyperbolic geometry.
   - (D) less than $2\pi$ in non-Euclidean geometry and equal to $2\pi$ in Euclidean geometry.

C

   (b) Which of these is **not** a theorem of Hilbert geometry (without a parallel postulate)?
   - (A) If D is in the interior of $\angle$CAB, then so is every other point on $\overrightarrow{\mathrm{AD}}$ except A.
   - (B) Every point D in the interior of $\angle$CAB lies on a segment joining a point E on $\overrightarrow{\mathrm{AB}}$ to a point F on $\overrightarrow{\mathrm{AC}}$.
   - (C) If D lies on line $\overleftrightarrow{\mathrm{BC}}$, then D is in the interior of $\angle$CAB if B * D * C.
   - (D) If D lies on line $\overleftrightarrow{\mathrm{BC}}$, then B * D * C if D is in the interior of $\angle$CAB.
   - (E) If $\overrightarrow{\mathrm{AD}}$ is between $\overrightarrow{\mathrm{AC}}$ and $\overrightarrow{\mathrm{AB}}$, then $\overrightarrow{\mathrm{AD}}$ intersects segment BC.

B (“Warning”, p. 115)

   (c) In the right-hand side of the Poincaré cross-ratio formula, the distance from A to P in the Poincaré (hyperbolic) metric along a Poincaré geodesic. the notation ($\overline{\mathrm{AP}}$) means
   - (A) the distance from A to P in the Poincaré (hyperbolic) metric along a Poincaré geodesic.
   - (B) the distance from A to P in the Poincaré metric along a Euclidean geodesic (straight line).
   - (C) the Euclidean (arc length) distance from A to P along a Poincaré geodesic.
   - (D) the Euclidean distance from A to P along a straight line.

D

   (d) The **converse** of the alternate interior angle theorem
   - (A) is equivalent (in Hilbert's neutral geometry) to the Euclidean parallel postulate.
   - (B) was first proved by Legendre.
   - (C) shows that the elliptic parallelism property is inconsistent with Hilbert's axioms.
   - (D) is satisfied by Dehn's models, because they have very few parallel lines.
   - (E) shows that Saccheri's “hypothesis of the obtuse angle” is never true.

A

2. *(30 pts.)* Rearrange these names into historical order, earliest to latest:

   Bolyai, Euclid, Klein, Saccheri, Thales, Proclus

Thales, Euclid, Proclus, Saccheri, Bolyai, Klein

3. *(20 pts.)* Prove that in a Saccheri quadrilateral, the line joining the midpoints of the summit and the base is perpendicular to both the summit and the base. (You can take it as known that the summit angles of a Saccheri quadrilateral are congruent.)

[See pp. 177-178.]

4. *(20 pts.)* In the Poincaré disk model, the formula for arc length is $ds^2 = \frac{4(dx^2+dy^2)}{[1-(x^2+y^2)]^2}$. Introduce polar coordinates $(\rho, \theta)$ in the $(x, y)$ plane and show that the further coordinate transformation $\rho = \tanh\left(\frac{r}{2}\right)$ converts the arc length to $ds^2 = dr^2 + \sinh^2 r\, d\theta^2$. Explain why this result justifies the vague claim that hyperbolic geometry describes "a sphere of imaginary radius".

$$ds^2 = \frac{4(d\rho^2 + \rho^2 d\theta^2)}{(1-\rho^2)^2}\,.$$

$$d\rho = \frac{1}{2}\operatorname{sech}^2\left(\frac{r}{2}\right) dr \qquad \text{(see below);}$$

so

$$ds^2 = \frac{4\left(\frac{1}{4}\operatorname{sech}^4\frac{r}{2}\, dr^2 + \tanh^2\frac{r}{2}\, d\theta^2\right)}{\operatorname{sech}^4\frac{r}{2}} = dr^2 + \frac{4\tanh^2\frac{r}{2}}{\operatorname{sech}^4\frac{r}{2}}\,.$$

But

$$\sinh 2z = 2\sinh z\cosh z\,,$$

so

$$\sinh^2 r = \frac{4\sinh^2\frac{r}{2}}{\operatorname{sech}^2\frac{r}{2}} = \frac{4\tanh^2\frac{r}{2}}{\operatorname{sech}^4\frac{r}{2}}\,.$$

This proves the claim. Changing $\theta$ to $i\theta$ changes trig functions to hyperbolic ones and hence sphere formulas to hyperboloidal ones.

Let's check that I remembered the hyperbolic identities correctly:

$$\tanh z = \frac{\sinh z}{\cosh z} \;\Rightarrow\; \tanh' z = \frac{\cosh^2 z - \sinh^2 z}{\cosh^2 z} = \begin{cases} \frac{1}{\cosh^2 z} \equiv \operatorname{sech}^2 z\,, \\ 1 - \tanh^2 z\,. \end{cases}$$

$$2\sinh z\cosh z = \frac{1}{2}(e^z - e^{-z})(e^z + e^{-z}) = \frac{1}{2}(e^{2z} - e^{-2z}) = \sinh 2z\,.$$

Generally speaking, every trig identity gives a hyperbolic identity that is correct up to a possible minus sign somewhere.

5. *(15 pts.)* Explain the statement:

   If you succeed in proving the parallel postulate, you will have shown that Euclidean geometry is *inconsistent*.

[See notes, beginning of Chapter 7.]

6. *(20 pts.)* Prove the ASA criterion for triangle congruence: If $\angle$ A $\cong$ $\angle$ D, $\angle$ C $\cong$ $\angle$F, and AC $\cong$ DF, then $\triangle$ABC $\cong$ $\triangle$DEF.

[See Exercise 3.26.]

7. *(Essay – 25 pts.)* In this course we studied five models of 2-dimensional hyperbolic geometry. List them, and describe the principal properties of each. Discuss their relative advantages and disadvantages. (You will probably have one paragraph for each model, and then some more paragraphs discussing more than one model, contrasting their advantages.)

[The 5 models intended are those listed on pp. 48–49 of notes, which satisfy the Hilbert axioms including Dedekind and thus embody the "standard" hyperbolic plane (except that the pseudosphere is only local). Some students listed a Dehn model (which doesn't satisfy Dedekind's axiom) or the 5-point incidence plane from Chapter 2 (which doesn't satisfy the B and C axioms).]