Math. 467, Sec. 901 (Fulling) 14 December 2016

Final Examination – Solutions

Name: ____________________

1. *(Multiple choice – each 5 pts.)* Circle the correct capital letter.

(a) The ratio of a circle's circumference to its radius is
- (A) always equal to 2π.
- (B) greater than 2π in elliptic geometry and less than 2π in hyperbolic geometry.
- (C) less than 2π in elliptic geometry and greater than 2π in hyperbolic geometry.
- (D) less than 2π in non-Euclidean geometry and equal to 2π in Euclidean geometry.

C

(b) Which of these is **not** a theorem of Hilbert geometry (without a parallel postulate)?
- (A) If D is in the interior of $\angle$CAB, then so is every other point on $\overrightarrow{\text{AD}}$ except A.
- (B) Every point D in the interior of $\angle$CAB lies on a segment joining a point E on $\overrightarrow{\text{AB}}$ to a point F on $\overrightarrow{\text{AC}}$.
- (C) If D lies on line $\overleftrightarrow{\text{BC}}$, then D is in the interior of $\angle$CAB if B * D * C.
- (D) If D lies on line $\overleftrightarrow{\text{BC}}$, then B * D * C if D is in the interior of $\angle$CAB.
- (E) If $\overrightarrow{\text{AD}}$ is between $\overrightarrow{\text{AC}}$ and $\overrightarrow{\text{AB}}$, then $\overrightarrow{\text{AD}}$ intersects segment BC.

B ("Warning", p. 115)

(c) A model of 2-dimensional hyperbolic geometry that can be realized by a physical object in 3-dimensional Euclidean space is
- (A) the Klein disk.
- (B) the Poincaré disk.
- (C) the Beltrami pseudosphere
- (D) the hyperboloid.

C

(d) Which of these remarks about "models" is **NOT** true?
- (A) Models of the Hilbert I axioms exist with finitely many points.
- (B) Models of the Hilbert B axioms require infinitely many points.
- (C) Models are crucial in proving the independence of the parallel postulate.
- (D) In a good model the basic terms of a theory may not be reinterpreted in unexpected ways.

D

(e) In "neutral geometry" it is possible to prove
- (A) Parallel lines exist.
- (B) The sum of the angles in a triangle is 180°.
- (C) If a line intersects one of two parallel lines, then it intersects the other.
- (D) Rectangles exist.

A

(f) Poincaré's disk model is better than Klein's in which respect?
  (A) It satisfies more of the axioms.
  (B) It is *conformal*, meaning that it represents angles accurately.
  (C) Its lines are straight Euclidean lines.
  (D) [none of these]

B

(g) If the "acute angle hypothesis" holds, which of these does **NOT** follow?
  (A) A side opposite to the fourth angle of a Lambert quadrilateral is greater than the adjacent side.
  (B) The fourth angle of every Lambert quadrilateral is acute.
  (C) The angle sum of a triangle is less than two right angles.
  (D) Rectangles do not exist.

A

(h) The space with arc length formula $ds^2 = e^{2x}\,dx^2 + dy^2$
  (A) has negative curvature, because the factor e^{2x} is concave upward.
  (B) has positive curvature, because the functions multiplying dx^2 and dy^2 have the same sign.
  (C) is flat, because the coordinate transformation $x = \ln u$ converts it to $ds^2 = du^2 + dy^2$.
  (D) has different curvature at different points, because e^{2x} is a nontrivial function of x.

C

2. *(30 pts.)* Rearrange these names into historical order, earliest to latest:

   Lobachevsky, Euclid, Hilbert, Saccheri, Pythagoras, Archimedes

Pythagoras, Euclid, Archimedes, Saccheri, Lobachevsky, Hilbert

3. *(20 pts.)* Suppose that A and B are points in the Poincaré disk and that P and Q (points on the circle bounding the disk) are the ends of the Poincaré line through A and B. The *cross-ratio* is defined as

$$(\mathrm{AB}, \mathrm{PQ}) \equiv \frac{\overline{\mathrm{AP}}}{\overline{\mathrm{AQ}}}\,\frac{\overline{\mathrm{BQ}}}{\overline{\mathrm{BP}}}\,,$$

   where, for instance, $\overline{\mathrm{AB}}$ is the *Euclidean* distance between A and B. Show how to use the cross-ratio to define a distance (length) function in the hyperbolic geometry that is *additive* in the sense that $d(\mathrm{AB}) = d(\mathrm{AC}) + d(\mathrm{CB})$ when $\mathrm{A} * \mathrm{C} * \mathrm{B}$ along a Poincaré line. (Verify that it is indeed additive.)

[See book, pp. 319–321.]

4. *(15 pts.)* Explain the statement:

   If you succeed in proving the parallel postulate, you will have shown that Euclidean geometry is *inconsistent*.

[See notes, beginning of Chapter 7.]

5. *(20 pts.)* Prove the SSS criterion for triangle congruence: If AB $\cong$ DE, BC $\cong$ EF, and AC $\cong$ DF, then $\triangle$ABC $\cong$ $\triangle$DEF.

6. *(Essay – 25 pts.)* State the Alternate Interior Angle Theorem **and** its converse. Discuss them and their significance. In particular: What assumptions (axioms) are needed to prove each one? What is the significance of each in regard to the issue of parallelism (elliptic, Euclidean, or hyperbolic)?