Math. 467 (Fulling) 2 December 2020

Final Examination – Solutions

Upload your answers, in order, as a single document.

1. *(Multiple choice – each 5 pts.)*

(a) The curvature of a surface is
 (A) negative for a sphere and positive for a hyperboloid.
 (B) negative when circumferences are smaller than 2π times radii and positive when angle sums (of triangles) are greater than 180°.
 (C) positive when circumferences are smaller than 2π times radii and negative when angle sums are less than 180°.
 (D) positive when circumferences are smaller than 2π times radii and negative when angle sums are greater than 180°.

C

(b) Historically proposed axioms (or tacit assumptions) equivalent to the parallel postulate (HE) include all of the following **EXCEPT**
 (A) Rectangles exist.
 (B) An equidistant locus is the same thing as a parallel line.
 (C) Limiting parallel rays (without a common perpendicular) exist.
 (D) Geometry is the same at all scales.

C (It holds in hyperbolic geometry and is inconsistent with HE.)

(c) Dehn's models show that
 (A) geometries that do not satisfy Dedekind's axiom are inconsistent.
 (B) hyperbolic parallelism is not exactly equivalent to the acute angle hypothesis.
 (C) elliptic parallelism is consistent with the Hilbert IBC axioms.
 (D) if the sum of the angles in a triangle is always 180°, then Euclid's parallel postulate holds.

B

(d) Numerical measurement of angles and segments (e.g., degrees and meters)
 (A) is impossible in Euclidean geometry, because of Zeno's paradoxes.
 (B) is possible only if the parallel postulate is assumed.
 (C) is contrary to the spirit of Euclid's approach, but nevertheless is perfectly rigorous from a modern point of view.
 (D) became possible only after Hilbert introduced the real numbers into geometry.

C

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2. *(30 pts.)* Rearrange these names into historical order, earliest to latest:

Saccheri, Thales, Euclid, Bolyai (the father) , Poincaré, Archimedes

Thales, Euclid, Archimedes, Saccheri, Bolyai, Poincaré

3. *(20 pts.)*

(a) State the **exterior angle theorem** (the version that is a theorem in neutral geometry).

[See Greenberg p. 164.]

(b) State the **stronger** version of the exterior angle theorem that is true in Euclidean geometry.

An exterior angle is congruent to the sum of the two remote interior angles [p. 176].

(c) Each of those two theorems is equivalent (respectively) to a certain theorem about a sum of **interior** angles of an arbitrary triangle. State those two angle-sum theorems.

The Euclidean theorem is that the sum of all three angles is 180° (if the parallel postulate holds) [see p. 175].

In the neutral setting, the technically correct answer is that the sum of any *two* angles of a triangle is less than 180° [p. 171]. A more interesting conclusion that is not quite "equivalent" is that if Aristotle's or Archimedes' axiom holds, then the sum of all three angles is $\leq 180^\circ$ [pp. 185–186].

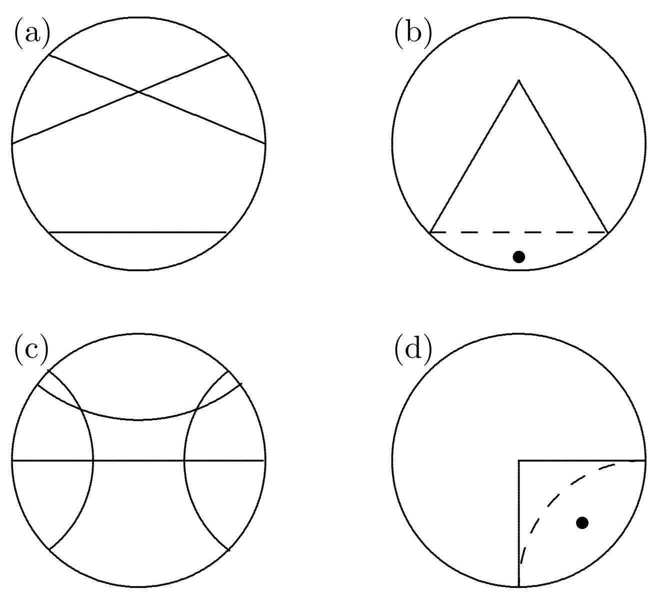
4. *(20 pts.)*

(a) In the Klein disk, draw a sketch showing how the parallel postulate (HE) can be violated.

(b) In the Klein disk, draw a sketch showing how a point can be inside an angle but not connected to the sides of the angle by any line.

(c) In the Poincaré disk, draw 4 Poincaré lines that bound a Saccheri quadrilateral.

(d) In the Poincaré disk, sketch the counterpart of the situation in (b).



5. *(15 pts.)*

(a) State the **converse** of the alternate interior angle theorem, including all essential hypotheses.

If the parallel postulate holds, then when parallel lines are crossed by a third line, the alternate interior angles are equal.

(b) Discuss the significance of this theorem (the converse of AIA) in the context of the parallel postulate and hyperbolic geometry.

The converse of AIA is equivalent to the parallel postulate (given the Hilbert IBC axioms) [p. 175] and therefore is not true in hyperbolic geometry. (In contrast, the direct AIA does not need the parallel postulate for its proof, but it does require some IBC axioms that are violated by elliptic geometry, and in fact it is not true in elliptic geometry.)

6. *(20 pts.)* Prove **ONE** of the famous triangle congruence theorems, ASA or SSS. **If you try both, clearly indicate which one you want graded.**

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7. *(Essay – 25 pts.)*

This course has centered on the Euclidean parallel postulate. Your essay should explain what the postulate is (concentrate on the Proclus–Playfair form), explain its logical status (proved? refuted? whatever?), and discuss the historical and philosophical significance of this issue.