Math. 467 (Fulling) 11 May 2011

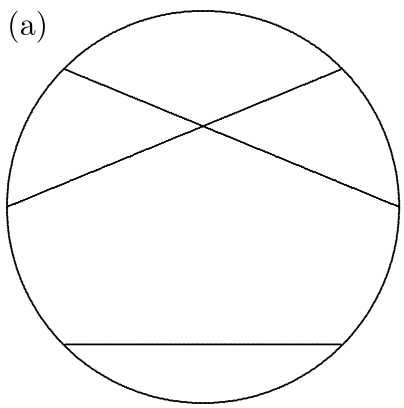
Final Examination – Solutions

Name: ______________________________

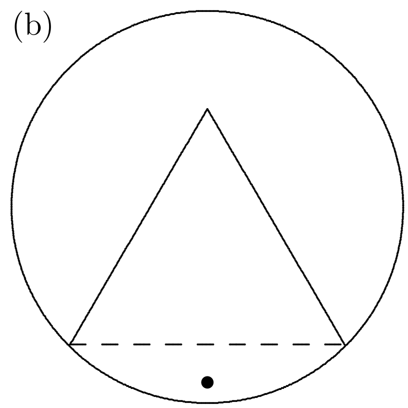
1. *(50 pts.)* was the take-home essay.

2. *(20 pts.)*
   (a) In the Klein disk, draw a sketch showing how the parallel postulate (HE) can be violated.
   (b) In the Klein disk, draw a sketch showing how a point can be inside an angle but not connected to the sides of the angle by any line.
   (c) In the Poincaré disk, draw 4 Poincaré lines that bound a Saccheri quadrilateral.
   (d) In the Poincaré disk, sketch the counterpart of the situation in (b).

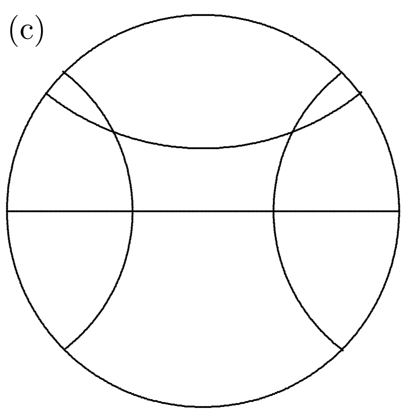
(a)



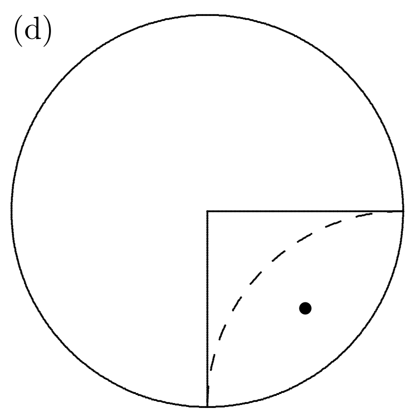
(b)



(c)



(d)



(In (d) the rays are straight because the vertex is at the center.)

3. *(20 pts.)* Rearrange these names into historical order, earliest to latest:

Pythagoras, Dehn, Bolyai (the son), Euclid, Lambert

Pythagoras, Euclid, Lambert, Bolyai, Dehn

4. *(Multiple choice – each 5 pts.)*

(a) The search for a proof of Euclid's postulate "EV"
(A) is still going on.
(B) ended successfully.
(C) ended unsuccessfully (with a *dis*proof).
(D) was rendered pointless by the discovery of hyperbolic geometry.

D

(b) The ratio of a circle's circumference to its radius is
(A) always equal to 2π.
(B) greater than 2π in elliptic geometry and less than 2π in hyperbolic geometry.
(C) less than 2π in elliptic geometry and greater than 2π in hyperbolic geometry.
(D) less than 2π in non-Euclidean geometry and equal to 2π in Euclidean geometry.

C

(c) The spherical geometry of the earth's surface (idealized) is inconsistent with the Hilbert axioms because of all the following reasons **EXCEPT**
(A) It contradicts the alternate interior angle theorem.
(B) It violates Axiom B-3.
(C) It violates Axiom B-4 (a line divides space into two sides).
(D) It violates Axiom I-1 (two points determine a line).

C

(d) A model of hyperbolic geometry that can be literally realized by a 3-dimensional object is
(A) Poincaré's upper half plane.
(B) Poincaré's disk.
(C) Klein's disk.
(D) Beltrami's pseudosphere.

D

(e) Klein's disk model is better than Poincaré's in which respect?
(A) It satisfies more of the axioms.
(B) It is *conformal*, meaning that it represents angles accurately.
(C) Its lines are straight Euclidean lines.
(D) [none of these]

C

(f) The *defect* of a triangle
(A) was defined by Greenberg when he first started talking about it.
(B) is proportional to the area of the triangle.
(C) is negative in hyperbolic geometry.
(D) is zero in elliptic geometry.

B

(g) Historically proposed axioms (or tacit assumptions) equivalent to the parallel postulate (HE) include all of the following **EXCEPT**

(A) Rectangles exist.
(B) An equidistant locus is the same thing as a parallel line.
(C) Limiting parallel rays (without a common perpendicular) exist.
(D) Geometry is the same at all scales.

C (HH) *contradicts* HE. (A = Clairaut, B = Clavius, D = Wallis)

(h) Which of these is true in hyperbolic geometry?

(A) An exterior angle of a triangle is greater than either remote interior angle.
(B) Two lines parallel to the same line are parallel to each other.
(C) the obtuse angle hypothesis.
(D) the converse of the alternate interior angle theorem.

A

5. *(20 pts.)* Recall that the *Line–Circle Continuity Principle* states: If a line l passes through a point P inside a circle γ, then l intersects γ in two points, say A and B. Intuitively it is "obvious" that P lies *between* A and B, but the principle as stated does not demand this. **Prove** that if the line–circle continuity principle holds, then indeed A $*$ P $*$ B . Your proof should be valid in neutral geometry, so it should not use the statement that the angle sum of a triangle is 180°.

Assume to the contrary that A $*$ B $*$ P . (The case P $*$ A $*$ B can be handled in the same way, and the cases P = A and P = B are impossible since P is not *on* the circle.) Let O be the center of γ. The angle $\angle$ABO is an exterior angle to triangle ΔBPO, hence greater than $\angle$BPO (= $\angle$APO). The triangle AOB is isosceles, so $\angle$ABO = $\angle$BAO (= $\angle$PAO). Therefore, since "the greater angle is opposite the greater side" in ΔAPO, segment OP is greater than segment OA. This means that P is *outside* the triangle, contrary to assumption.

(Variant argument: By "Corollary 2 to the EA theorem" the sum of any two angles of a triangle is less than 180°. Therefore, the base angles of an isosceles triangle must be acute, and thus $\angle$PBO is obtuse and we can apply "the greater angle . . . " directly to ΔPBO. Note that appealing to the Saccheri–Legendre theorem or the non-obtuse angle theorem is less satisfactory, since they require the Archimedes or the Aristotle axiom.)

6. *(10 bonus pts.)* Into the (Euclidean) Cartesian plane with standard coordinates (x, y), introduce new coordinates (u, v) by the equations $x = u^3$, $y = 2v$. Suppose that a curve is presented to us by giving the coordinates of its points as functions, $\big(u(t), v(t)\big)$, of a parameter, t. What is the formula (an integral) for the length of the part of this curve between the points labeled by two parameter values, t_1 and t_2 ? (Let $\dot{u} = \frac{du}{dt}$, etc.)

$$dx = 3u^2\, du, \quad dy = 2, dv.$$

$$ds^2 = dx^2 + dy^2 = 9u^4\, du^2 + 4\, dv^2.$$

Therefore,

$$s = \int_{t_1}^{t_2} \sqrt{9u^4\dot{u}^2 + 4\dot{v}^2}\, dt.$$