Math. 467 (Fulling) 7 May 2013

Final Examination – Solutions

Name: ____

1. *(40 pts.)* was the take-home essay.

2. *(Multiple choice – each 5 pts.)*

   (a) Which of these is true in hyperbolic geometry?
   (A) An exterior angle of a triangle is equal to the sum of the two remote interior angles.
   (B) Two lines parallel to the same line are parallel to each other.
   (C) the acute angle hypothesis.
   (D) the converse of the alternate interior angle theorem.

C

   (b) The ratio of a circle's circumference to its radius is
   (A) greater than 2π in elliptic geometry and less than 2π in hyperbolic geometry.
   (B) less than 2π in elliptic geometry and greater than 2π in hyperbolic geometry.
   (C) less than 2π in non-Euclidean geometry and equal to 2π in Euclidean geometry.
   (D) always equal to 2π.

B

   (c) The property of Beltrami's pseudosphere model that makes it unique is
   (A) Its lines are actually straight lines in the Euclidean sense.
   (B) It can be realized by a 3-dimensional object in physical Euclidean space.
   (C) It represents angles accurately (conformally).
   (D) It represents the entire hyperbolic plane, not just a portion of it.

B

   (d) $p \Rightarrow q$ is equivalent to
   (A) $p \vee \neg q$
   (B) $q \vee \neg p$
   (C) $\neg p \Rightarrow \neg q$
   (D) $\neg(q \wedge \neg p)$

B

   (e) In "neutral geometry" it is possible to prove
   (A) Rectangles exist.
   (B) The sum of the angles in a triangle is 180°.
   (C) If a line intersects one of two parallel lines, then it intersects the other.
   (D) Parallel lines exist.

D (corollary to AIA). The others are strictly Euclidean (for C, see Prop. 4.7).

3. *(20 pts.)* Rearrange these names into historical order, earliest to latest:

   Lobachevsky, Poincaré, Archimedes, Thales, Saccheri

Thales, Archimedes, Saccheri, Lobachevsky, Poincaré

4. *(30 pts.)*

   (a) Define "Saccheri quadrilateral".

[See pp. 176–177.]

   (b) Prove that the summit angles of a Saccheri quadrilateral are congruent to each other.

   (c) Prove that the line joining the midpoints of the summit and the base of a Saccheri quadrilateral is perpendicular to both the summit and the base.

[(b) and (c) are "Saccheri I and II", also known as Prop. 4.12. The proofs (p. 178) are straightforward exercises with congruent triangles.]

5. *(Essay – 15 pts.)* Explain the statement:

   If you succeed in proving the parallel postulate, you will have shown that Euclidean geometry is *inconsistent*.

[See Greenberg pp. 289–293 and 72–79 and related pages of my lecture notes.]

6. *(10 bonus pts.)* An applicant for an NSF grant proposes to study a new kind of 2-dimensional geometry defined by the arc length formula

$$ds^2 = e^{2x}\,dx^2 + y^4\,dy^2\,.$$

   Show that the space in question is nothing but a portion of the ordinary Euclidean plane in disguise. (Define some new coordinates that make that obvious.)

An arc-length expression of the form $f(x)^2\,dx^2$ can be converted to dX^2 by demanding that $dX = f(x)\,dx$, which means that $X(x)$ is any antiderivative of f. In our case we have

$$dX = e^x\,dx \;\Rightarrow\; X = e^x \;\Rightarrow\; x = \ln X\,,$$

$$dY = y^2\,dy \;\Rightarrow\; Y = \frac{1}{3}y^3 \;\Rightarrow\; y = \sqrt[3]{3Y}\,.$$

(Since any antiderivative will do, we can ignore constants of integration.) Then

$$ds^2 = dX^2 + dY^2\,,$$

so the geometry is Euclidean. But we get only half of the plane, because X ranges from 0 to ∞ as x goes from $-\infty$ to ∞.

7. *(20 pts.)* State and prove **ONE** of these. **If you try more than one, clearly indicate which one you want graded.** (Use separate paper, which will be provided.)

   (A) ASA congruence criterion
   (B) SSS congruence criterion
   (C) Exterior angle theorem
   (D) Hypotenuse–leg congruence criterion

[These proofs are all in the book or the homework.]