Math. 467 (Fulling) 1 March 2013

Midterm Test – Solutions

Name: ______________________________

1. *(Multiple choice – each 5 pts.) (Circle the correct capital letter.)*

   (a) In the Hilbert approach, which triangle congruence theorem becomes an axiom?

   (A) none
   (B) SSS
   (C) SAS
   (D) ASA
   (E) all

C

   (b) Euclid's geometry is based on five fundamental, undefined terms, but the importance of one of them was not fully realized until the work of Pasch and Hilbert over 2000 years later. Which one?

   (A) point
   (B) line
   (C) lie on
   (D) between
   (E) congruent

D

   (c) Which of these is **not** a theorem of Hilbert geometry (without a parallel postulate)?

   (A) If D is in the interior of $\angle$CAB, then so is every other point on $\overrightarrow{\mathrm{AD}}$ except A.
   (B) Every point D in the interior of $\angle$CAB lies on a segment joining a point E on $\overrightarrow{\mathrm{AB}}$ to a point F on $\overrightarrow{\mathrm{AC}}$.
   (C) If D lies on line $\overleftrightarrow{\mathrm{BC}}$, then D is in the interior of $\angle$CAB if B * D * C.
   (D) If D lies on line $\overleftrightarrow{\mathrm{BC}}$, then B * D * C if D is in the interior of $\angle$CAB.
   (E) If $\overrightarrow{\mathrm{AD}}$ is between $\overrightarrow{\mathrm{AC}}$ and $\overrightarrow{\mathrm{AB}}$, then $\overrightarrow{\mathrm{AD}}$ intersects segment BC.

B (This is the "WARNING" on p. 115.)

   (d) By definition, lines are *parallel* if

   (A) They are a constant distance apart.
   (B) They divide the plane into three "sides".
   (C) Each of them divides the plane into two "sides".
   (D) They do not intersect.
   (E) There is a unique line perpendicular to both of them.

D

2. *(10 pts.)* Simplify $\neg\exists x\,\forall y\,[(x \le y) \;\Rightarrow\; [x = y \vee \exists z\,(x < z < y)]]$.
(Push the "$\neg$" in as far as you can! Truth or falsity of the statement is irrelevant. In Greenberg's notation, $\neg$ is $\sim$.)

Let's reduce the expression in steps, using various De Morgan rules.

$$\forall x\,\exists y\,\neg[(x \le y) \;\Rightarrow\; [x = y \vee \exists z\,(x < z < y)]].$$

$$\forall x\,\exists y\,[(x \le y) \wedge \neg[x = y \vee \exists z\,(x < z < y)]]$$

(remember the truth table for " $\Rightarrow$ ", or Logic Rule 4, p. 61).

$$\forall x\,\exists y\,[(x \le y) \wedge [\neg(x = y) \wedge \neg\exists z\,(x < z < y)]].$$

$$\forall x\,\exists y\,[(x < y) \wedge \forall z\,\neg(x < z < y)].$$

(The first simplification in the last line is not mandatory, but it is common sense, so let's make it.) Now note that $x < z < y$ is short for $x < z \wedge z < y$, so when we negate it we get

$$\forall x\,\exists y\,[(x < y) \wedge \forall z\,(x \ge z \vee z \ge y)].$$

3. *(20 pts.)* Carefully state (don't prove) these theorems. Illustrate each with a diagram.
   (A) Pasch's theorem
   (B) The crossbar theorem

See pp. 114 and 116.

4. *(10 pts.)* Explain what the term *partition* means (as used in the theorem "Every equivalence relation determines a partition and vice versa.").

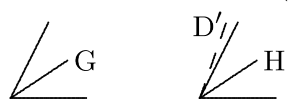
A partition is a way of dividing up a set into parts that are *exclusive* and *exhaustive*. More precisely, a partition of a set S is a collection $\{S_i\}$ of subsets of S such that:

- The subsets are (pairwise) disjoint: $S_i \cap S_j = \emptyset$ if $i \ne j$.
- The subsets exhaust S: $S = \bigcup_i S_i$.

5. *(16 pts.)* Prove **ONE** of these: *(At most 8 points extra credit for* **both**.*)*
   (A) For segments, if $AB < CD$, then $2AB < 2CD$. (Here $2AB$ means the result of adding AB to a copy of itself, in accordance with Axiom C-3.)

This is Exercise 3.33. Paraphrasing from the Instructor's Manual: Without loss of generality, assume $B = D$ and $C * A * B$. Choose A' on the ray opposite $\overrightarrow{BC}$ so that $AB \cong BA'$, and choose C' analogously; then $AA' = 2AB$ and $CC' = 2BC$. Now $BA' < BC'$ (Prop. 3.13, since $BA < BC$), so $B * A' * C'$. Thus $C * A * A' * C'$ (Prop. 3.3). So apply the definition of "$<$" twice: $AA' < CA' < CC'$.

   (B) Given $\overrightarrow{BG}$ between $\overrightarrow{BA}$ and $\overrightarrow{BC}$, and $\overrightarrow{EH}$ between $\overrightarrow{ED}$ and $\overrightarrow{EF}$, if $\angle CBG \cong \angle FEH$ and $\angle ABC \cong \angle DEF$, then $\angle GBA \cong \angle HED$. (This is Proposition 3.20, "angle subtraction". You are allowed to use the preceding proposition (3.19), the analog for angle addition.)



Again from Instructor's Manual: By C-4 there is a ray $\overrightarrow{ED'}$ on the opposite (from F) side of $\overleftrightarrow{EH}$ with $\angle ABG \cong \angle D'EH$. By Prop. 3.19, $\angle ABC \cong \angle D'EF$. By the uniqueness clause in C-4, $\overrightarrow{ED'}$ must be the same as $\overrightarrow{ED}$.

6. *(Essay – 24 pts.)* Recall the three incidence axioms (two expressed in logical symbols to save space):

   1. $\forall P \forall Q\,[(P \neq Q) \;\Rightarrow\; \exists! l\,(P \;\mathtt{I}\; l \,\wedge\, Q \;\mathtt{I}\; l)]$
   2. $\forall l \exists P \exists Q\,[P \neq Q \wedge (P \;\mathtt{I}\; l \,\wedge\, Q \;\mathtt{I}\; l)]$
   3. There exist 3 distinct points that are not collinear.

   (You don't need to repeat these axioms in your essay.)

   Describe the models (each consisting of finitely many points) that demonstrate that these axioms are consistent with any of the three parallelism properties (*Euclidean*, *elliptic*, and *hyperbolic*, which you should define in passing). Complete proofs are not required, but brief explanations never hurt, especially if partial credit becomes an issue.