Math. 467: Modern Geometry
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First day

Course handout:

> http://calclab.math.tamu.edu/~fulling/m467/s17/handout.pdf

Class web page:

> http://calclab.math.tamu.edu/~fulling/m467/s17/

[Let's look at them on the screen, along with the first assignment page, e-campus, and other links.]

Please introduce yourself (on paper)**:**

Name, major, class

Future plans (career, etc.)

Why are you here? What do you expect to get out of the course?

Content and objectives of the course

This is a W course. There is more emphasis on homework and term papers and less on tests than in most math courses. Each homework assignment will contain one essay question that will be graded by me personally and returned to you for revision. The same goes for the two term papers and one question on the midterm test.

[Refer to handout, top half p. 1 and p. 3.] Chapters 7 and 8 are covered superficially.

Recent changes:

1. Prerequisites: No real change, just a nuance. The purpose of the linear algebra prerequisite is to ensure some experience with proofs. Any linear algebra course (Math. 304, 309, 311, 323) or Math. 220 will do.

2. As of Fall '16, the percentages of the grade devoted to class participation and homework essay questions are slightly increased. Therefore, I will be more careful about assessing the contribution of each team member to class presentations; more later about the procedure.

3. In accordance with university regulations: *To get W credit for the course, you must have a passing grade on the writing assignments, not just on the course as a whole.* (This was always true in theory, but it was not made explicit before.)

4. In term papers, please give "mathematics style" bibliographical citations. even if that's not what your English teacher told you to do.

Please read the handout thoroughly. Note that the first reading and homework assignment is on the Web, and the written homework is due next Friday. Skip the Preface but read the Introduction and Chapter 1 this week. Be prepared for discussion questions next Monday.

Later in the second week: Be prepared to form teams of 3 or 4.

Introduction

Geometry has two aspects:

1. A piece of abstract mathematics, closely related to the vector spaces $\mathbf{R}^2$ and $\mathbf{R}^3$ of linear algebra classes. (The conclusions follow from the definitions and assumptions.)

2. A theory of physical space, the world we live in. (This theory could be false, or only approximate. The world could be like a sphere, or like a crumpled sheet of paper.)

[Mention homogeneity and global topology.]

Axioms (or *postulates*) have been regarded in two ways:

1. "self-evident truths"

2. arbitrary starting assumptions

These don't exactly correlate with the previous list.

The parallel postulate is the arena where these issues were hashed out. Is it self-evident that (p. 21) "For every line l and every point P that does not lie on l, there exists a unique line m through P that is parallel to l."? [Sketch. Stimulate discussion.]

Definition: (p. 20) Two lines l and m are *parallel* if they do not intersect (even if extended).

The main subjects of our study:

Hilbert axioms — an updating of Euclid (for the plane).

Neutral geometry: all the theorems we can prove *without* the parallel postulate. This includes listing of statements *equivalent to* the parallel postulate, without judgment on their truth.

Euclidean geometry: plane geometry *with* the parallel postulate: exactly one parallel exists.

Hyperbolic geometry: more than one parallel. (This occupies the last part of the course.)

Elliptic geometry: no parallels at all! This is more problematical than hyperbolic (see Appendix A).

A *model* is an interpretation of the terms of a theory for which the truth of its axioms can be verified. (See p. 72.) Crudely speaking, one can find a model for each of the 3 geometries; this shows that the parallel postulate is independent of the other postulates.

Euclidean: $\mathbf{R}^2$.

Elliptic: The "lines" are great circles on a sphere. They are not infinite, and they intersect in *two* places, so we see that there will be complications.

Hyperbolic: a surface like a saddle or trumpet. (The circumference of a circle is greater than π times the diameter.)

What I have just said is a great oversimplification. Elliptic or spherical geometry is *not* a model of the Hilbert axioms, which is the main reason why we'll pay more attention to the hyperbolic case. (Furthermore, "elliptic" and "spherical" don't mean exactly the same thing.) Saddle and trumpet surfaces are not the whole hyperbolic geometry, just a fragment of it. All this will be explained in due course.

Chapter 1

[p. 1: Denounce the epigraph.]

ANCIENT MATHEMATICS

One sometimes hears that the ancient Hebrews believed that $\pi = 3$. [Discuss. Also the Indiana law.]

I Kings 7:23 : And he made a molten sea, ten cubits from the one brim to the other: it was round all about, ... and a line of thirty cubits did compass it round about.

GREEK MATHEMATICS

Three new elements (related):

1. proof (not just examples)

2. idealization (Lines are perfectly straight, infinitely thin, and (potentially) *infinitely long*, although Aristotle taught that the universe must be finite!)

3. math for math's sake (not primarily practical applications)

p. 4: Where did Greenberg get 6 ancient planets? I think he is counting the *celestial sphere*.

Euclid's Elements was the basic text for 2000 years. (T. L. Heath's early 20th-century translation and commentary constitute the standard English edition.) But see epigraph to Ch. 3, p. 103. Diagrams smuggle in assumptions; later we'll introduce *betweenness* concept and axioms to fill the gap.

The goal of this chapter is to give a quick modern cleanup and tour of Euclid's postulates. (I hope it's a review of high school geometry.) Part of the intention is to show that there are still ambiguities that make the rules of the game unclear, therefore motivating our later, slower work based on Hilbert's axioms. So, meanwhile, don't get too hot and bothered.

PRIMITIVE TERMS

(These also form the basis of the Hilbert theory.)

- point

- line

- lie on (as relation of point to line) (or "goes through" as relation of line to point)

- between (as relation of point to two other points)

- congruent (replacing ambiguous "equal"): having the same size and shape

Definitions

(These will remain in force in the Hilbert theory.)

- segment (between two points) (Includes endpoints.)

- circle (with given point as center and given segment as radius)

- rays and angles (see pp. 18–19). Note: To Greenberg,

  (1) An angle is a *set* of two rays; therefore, angle BAC = angle CAB automatically (no need for an axiom or theorem to that effect).

  (2) What we normally call 0° and 180°, and anything beyond (such as negative angles) are *not* angles for our present purposes.

- opposite rays; supplementary angles; right angles

- parallel (nonintersecting) lines. Recall that this presumes indefinite extendability; lines are not segments.

The 4 noncontroversial postulates

(These will *not* survive unscathed into the Hilbert formulation.)

I. Given two distinct points, P and Q, there is a unique line through them (i.e., P and Q lie on the line).

In Euclid this is stated as a *construction*: "To draw a straight line from any point to any point." In other words, "We shall use the straightedge as a basic tool."

III. Given distinct points O and A, there is a circle with center O and radius OA.

Again, in Euclid this is a construction: "To describe a circle with any center and any distance." In other words, "We shall use the compass as a basic tool."

II. For every segment AB and for every segment CD there exists a unique point E on line $\overleftrightarrow{AB}$ such that B is between A and E and segment CD is congruent to segment BE.

Euclid says, "To extend a finite straight line continuously in a straight line." In modern times it is often paraphrased as, "A straight line can be extended indefinitely in either direction," and the parallel postulate does not make much sense without this understanding. Presumably Greenberg has stated the postulate in the precise form that Euclid needs to use it later.

IV. All right angles are congruent.

Recall that an angle is defined to be a right angle if it has a supplementary angle to which it is congruent. We don't have "degrees" yet.

Euclid's version is the same, except for "equal to one another" instead of "congruent". But here is where the historical controversies began: Proclus argued that IV can be proved from the other postulates, and we'll see (pp. 128–129 and 143) that that is so in the Hilbert reformulation (which, however, has axioms that go beyond Euclid's first three).

<center>THE PARALLEL POSTULATE</center>

P. For every line l and every point P not on l, there is a unique line m through P parallel to l.

Euclid's fifth postulate was superficially very different:

E V. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Later geometers (notably Proclus (5th century) and Playfair (18th century)) settled on P as the best formulation of Euclidean parallelism.

Instructions and requests for W questions and term papers

1. Turn in W problems on separate paper (not stapled to the rest of the homework).

2. Use a word processor if possible. (Graphics software such as *Geometer's Sketchpad* is also laudable.) But I do not insist that you learn any particular software.

2a. If you have occasion to send me a paper electronically, please convert it to PDF or PostScript format (not `.doc`, for example).

3. It is not usually necessary in homework to copy the problem statement verbatim, but it's nice to make your essay self-contained; explain what you're talking about! On the major papers I insist on a descriptive title (not just "Exercise 3.2") and a decent introductory paragraph (don't just launch into a proof of something you haven't stated).

4. Guidelines for good writing:

   `http://calclab.math.tamu.edu/~fulling/m467/guidelines.pdf`

5. Double-spacing will be appreciated, especially on rough drafts. (Printing on both sides of the page is OK.)

6. Turn in the rough draft along with the finished product. The two should *not* be stapled together. Usually you won't get the rough draft back; it goes into my files.

7. Good writing does *not* mean:

   (1) Covers or binders.

   (2) Florid literary artistry (science fiction novels, for instance). Scientific writing is expected to be *sober* (which does not mean deadly dull). A *little* bit of humor may work.

   [Proceed to the class exercises and discussions from Chap 1.]