Proof of the Uniformity Theorem

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Lemma 1 (M.Ex. 4.5)

Look at Fig. 4.36. The big quadrilateral is Saccheri, so $\angle C \cong \angle D$ and $AC \cong BD$. The small quads are bi-right, so we can apply Prop. 4.13, "the greater angle is opposite the greater side". Since $\angle CPQ$ and $\angle DPQ$ are supplementary, either both are right angles, or one is acute and the other obtuse. Without loss of generality, assume that $\angle CPQ \le \angle DPQ$. Consider cases:

- $PQ > BD$: Then $\angle D > \angle DPQ \ge 90°$, so $\angle D$ and $\angle C$ are obtuse (c).
- $PQ < BD$: Then $PQ < AC$ and hence $\angle C < \angle CPQ \le 90°$. So $\angle C$ and $\angle D$ are acute (a).
- $PQ \cong BD$: Then all the quads are Saccheri, and hence all four summit angles are congruent. Since two of them are supplementary, they are all right angles. (b)

The converse assertions follow by the usual trichotomy argument.

Lemma 2 (M.Ex. 4.6)

First consider the case that $PQ \cong BD$. Then the big quad is Saccheri, and by Lemma 1(b) its summit angles are right. Therefore, so are the angles at D. (b)

Now suppose $PQ < BD$. There exists a point E such that $Q * P * E$ and $BD \cong QE$. Label the angles in Fig. 4.37, exploiting the fact that various quadrilaterals are Saccheri to equate three pairs of them (x, y, z). We want to show $u < x$ (so that x is obtuse). Well, by the exterior angle theorem, $w > v$. Also, looking at the angles at E, we see that $z < y$ (by Prop. 3.7 and $C * D * P$). Now

$$\begin{aligned} u &= z - w \quad \text{(angle subtraction)} \\ &< y - v \quad \text{(2 inequalities above)} \\ &= x \quad \text{(angle subtraction).} \end{aligned}$$

This proves (c).

Finally, let $PQ > BD$. Then there is a point E such that $P * E * Q$ and $QE \cong BD$. Draw a similar figure and label the angles similarly:

Again $w > v$ by the exterior angle theorem, but this time Prop. 3.7 (applied again at E) gives $y < z$. Therefore,

$$u = z + w > y + v = x \,.$$

Thus x is acute. (a)

As before, the converses follow by trichotomy.

The special case of congruent midlines (M.Ex. 4.7)

As in Fig. 4.38, we construct a replica of one quad on top of the other so that the midlines (and their feet) coincide. (The construction is unique because of uniqueness statements in many old axioms and theorems.) Midlines form right angles with both base and summit lines, so those lines also coincide for the two quads.

If CD = C'D', then the sides also coincide, since perpendiculars from a point are unique. Thus the two quads are identical and there is nothing to prove. So, without loss of generality we assume CD > C'D'. Hence C $*$ D' $*$ D. Consider cases:

BD > B'D': Then Lemma 2 $\Rightarrow$ the primed summit angles are acute, and Lemma 1 $\Rightarrow$ the unprimed summit angles are acute.

BD < B'D': Then Lemma 2 $\Rightarrow$ the primed summit angles are obtuse, and Lemma 1 $\Rightarrow$ the unprimed summit angles are obtuse.

BD $\cong$ B'D': Then Lemma 2 $\Rightarrow$ the primed summit angles are right, and Lemma 1 $\Rightarrow$ the unprimed summit angles are right.

So in any case the summit angles are all of the same type whenever Saccheri quadrilaterals have congruent midlines.

The general case (M.Ex. 4.8)

The proof is given in the book except for justifications of steps, which we can do as a class exercise. Note that the proof must be checked for the cases where N'B' $\geq$ NB, contrary to Fig. 4.39.