another point at which l meets γ (see Figure 4.35).

(b) Suppose now that l is tangent to γ at A. Prove that every point $B \neq A$ lying on l is outside γ, hence A is the unique point at which l meets γ. Prove that all points of γ other than A are on the same side of l. Prove conversely that if a line intersects a circle at only one point, then that line is tangent to the circle.

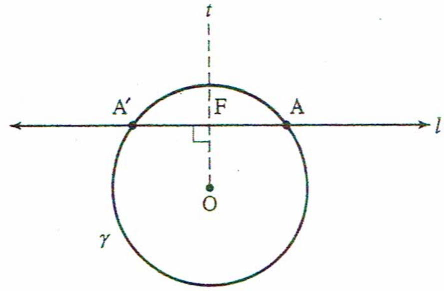


Figure 4.35

The next four major exercises provide a proof of the uniformity theorem. The first two are lemmas needed for the main argument in the third and fourth.

5. Prove Lemma 1. Given a Saccheri quadrilateral □ABDC and a point P between C and D. Let Q be the foot of the perpendicular from P to the base AB (Figure 4.36). Then
   (a) $PQ < BD$ iff the summit angles of □ABDC are acute.
   (b) $PQ \cong BD$ iff the the summit angles of □ABDC are right angles.
   (c) $PQ > BD$ iff the summit angles of □ABDC are obtuse.

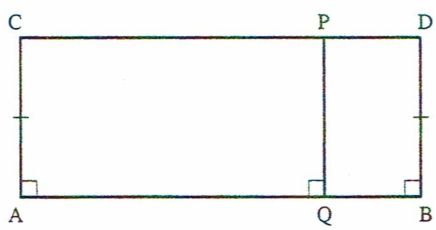


Figure 4.36

P such tha
P to $\overleftrightarrow{AB}$ (F
(a) PQ >]
(b) PQ ≅]
(c) PQ <]

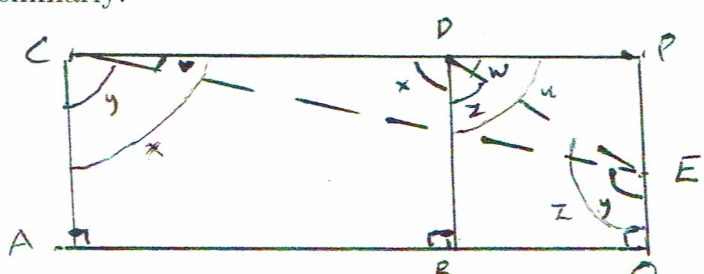
Figure 4.37

(Hints: Supp
(b) of Lemm
such that Q
quadrilateral
summit angl
∢BDP, imply
∢EDP is grea
implies ∢BD

Suppose
P ∗ E ∗ Q and
laterals □AQ
angles. The r
nally, show th
trichotomy.)

7. Prove the spe
segments of
(Figure 4.38).
for which M =

Finally, let $PQ > BD$. Then there is a point E such that $P * E * Q$ and $QE \cong BD$. Draw a similar figure and label the angles similarly:



(Hint: Apply Proposition 4.13 to the bi-right quadrilaterals □AQPC and □BQPD, using the fact that ∢QPC and ∢QPD are supplementary, the definition of a Saccheri quadrilateral, Proposition 4.12(a), and trichotomy.)

6. Prove Lemma 2. Given a Saccheri quadrilateral □ABDC and a point P such that C ∗ D ∗ P. Let Q be the foot of the perpendicular from P to $\overleftrightarrow{AB}$ (Figure 4.37). Then
   (a) PQ > BD iff the summit angles of □ABDC are acute.
   (b) PQ ≅ BD iff the summit angles of □ABDC are right angles.
   (c) PQ < BD iff the summit angles of □ABDC are obtuse.

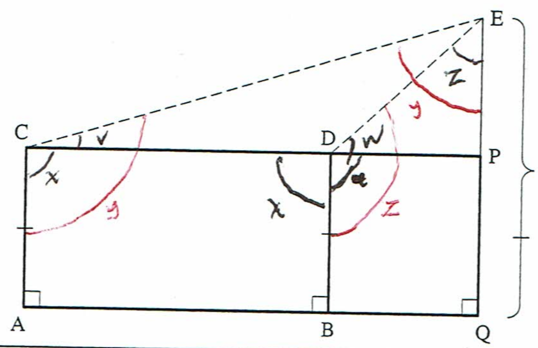


Figure 4.37

(Hints: Suppose PQ ≅ BD. Then □AQPC is Saccheri, so apply part (b) of Lemma 1. Suppose PQ < BD. Then there is a unique point E such that Q ∗ P ∗ E and QE ≅ BD. We then have two more Saccheri quadrilaterals □AQEC and □BQED, each of which has congruent summit angles. To show that ∢BDC is greater than its supplement ∢BDP, implying that it is obtuse, use the idea that exterior angle ∢EDP is greater than remote interior angle ∢ECD and that C ∗ D ∗ P implies ∢BDE < ∢ACE and subtract.

Suppose PQ > BD. Then there is a unique point E such that P ∗ E ∗ Q and QE ≅ BD. We again have two more Saccheri quadrilaterals □AQEC and □BQED, each of which has congruent summit angles. The rest of the argument is similar to the previous case. Finally, show that the other direction of these three cases follows from trichotomy.)

7. Prove the special case of the uniformity theorem where the midline segments of the two given Saccheri quadrilaterals are congruent (Figure 4.38). (Hints: First construct a congruent copy of □A′B′D′C′ for which M = M′ and N = N′, so we can assume these midpoints

On ray $\overrightarrow{NB}$, let H be the point such that $NH \cong N'B'$. On the same side of $\overleftrightarrow{LN}$ as H and on the perpendicular to $\overleftrightarrow{LN}$ through L, let G be the point such that $LG \cong M'D'$. Then $\triangle LNH \cong \triangle M'N'B'$, $\sphericalangle GLH \cong \sphericalangle D'M'B'$, $\triangle GLH \cong \triangle D'M'B'$, so that $\sphericalangle G \cong \sphericalangle D'$ and, by addition, $\sphericalangle NHG \cong \sphericalangle B'$, which is a right angle.

Since G and L lie on a parallel to $\overleftrightarrow{MD}$, they are on the same side of $\overleftrightarrow{MD}$, and since L is on the opposite side of $\overleftrightarrow{MD}$ from N, G is on the opposite side from H. Let K be the point at which GH meets line $\overleftrightarrow{MD}$, necessarily on ray $\overrightarrow{MD}$ since G and H are on the same side of $\overleftrightarrow{LN}$.

Now apply the important remark from the special case above: $\sphericalangle MKH$ is of the same type as $\sphericalangle D$. But $\sphericalangle MKH$ is also of the same type as $\sphericalangle G$. Therefore $\sphericalangle G$ and $\sphericalangle D$ are the same type of angle, and we are done. If $M'N' < MN$, reverse the roles. ◀

9. Denote by $|AB|$ the congruence class of segment AB (the set of all segments congruent to AB). Then, by definition and Axiom C-2,

$$AB \cong CD \Leftrightarrow |AB| = |CD|.$$

That is the underlying idea of "passing to the quotient"—replacing the equivalence relation $\cong$ with actual *equality* of equivalence classes.

We have already defined an ordering of segments: $AB < CD$ means that there is a point E between C and D such that $AB \cong CE$. (This seems to depend on the choice of one endpoint C of segment CD; show that it does not.) This ordering induces an ordering of segment congruence classes when we define

$$|AB| < |CD| \Leftrightarrow AB < CD.$$

This definition seems to depend on the choice of representatives of the equivalence classes; using Proposition 3.13, show that it is independent of that choice. Furthermore, show that Proposition 3.13 also yields the following information:

Trichotomy: $a < b$ or $a = b$ or $b < a$, and only one of these possibilities occurs.

Transitivity: $a < b$ and $b < c \Rightarrow a < c$.

Here a, b, c are arbitrary segment congruence classes.

We indicated in the discussion after the triangle inequality how to define addition of congruence classes. Show that Axiom C-3 guarantees that addition is well-defined.

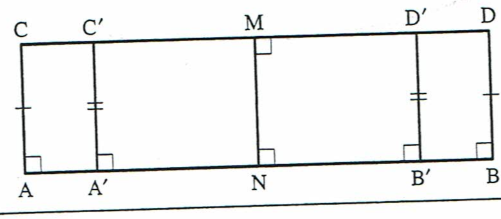


Figure 4.38

coincide, as do the summit and the base lines. Then apply the two lemmas in the preceding exercises.) **Important remark**: In this special case, we have also proved a uniformity result for Lambert quadrilaterals □MNBD and □MNB′D′, which have common side MN where there are right angles and common lines containing the sides adjacent to MN.

8. Here is a proof of the general case of the uniformity theorem from the three previous exercises. Your job is to provide justifications for the steps.

Proof:

The case MN ≅ M′N′ having been handled, consider the case M′N′ > MN. There is a unique point L such that L ∗ M ∗ N and LN ≅ M′N′. We will construct a Lambert quadrilateral □LNHG with the fourth angle at G, congruent to half of Saccheri quadrilateral □A′B′D′C′ (Figure 4.39).

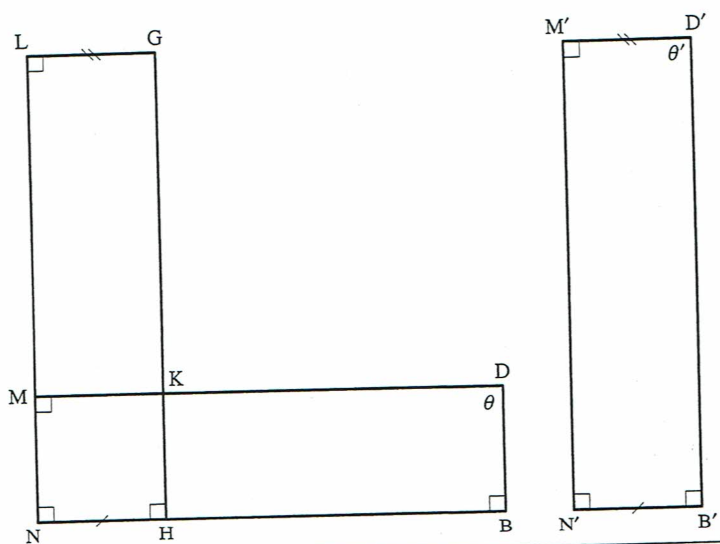


Figure 4.39