Math. 489GR (Fulling) 6 October 2005

Test A – Solutions

1. *(45 pts.)*

   (a) Let $L\colon \mathcal{V} \to \mathcal{V}$ be a linear operator mapping a vector space $\mathcal{V}$ into itself. Explain why L is called a $\binom{1}{1}$ tensor, and state the transformation law of the components of L (alias the matrix representing L) under a change of coordinates in $\mathcal{V}$ (alias a change of basis). (Define notation clearly.)

There are two lines of approach one could take here. In both cases we will eventually need a notation for coordinate transformations. Suppose that the components of vectors transform according to the formula

$$v^{\overline{\alpha}} = \Lambda^{\overline{\alpha}}{}_{\beta} v^{\beta}.$$

Then the components of one-forms transform by the contragredient formula,

$$p_{\overline{\alpha}} = \left(\Lambda^{-1}\right)^{\beta}{}_{\overline{\alpha}} \, p_{\beta} \, .$$

(I don't want to assume that Λ is a Lorentz transformation, so I don't write Λ^{-1} as Λ.)

Starting from the abstract: A $\binom{1}{1}$ tensor, by definition, is a bilinear mapping $L\colon \mathcal{V}^* \times \mathcal{V} \to \mathbf{R}$. That is, $L(\tilde{\omega}, \vec{v})$ is a number that depends linearly on each argument (variable), the one-form $\tilde{\omega}$ and the vector $\vec{v}$. If we apply the operator L to a vector, we get another vector, $L(\vec{v})$, and then for any one-form $\tilde{\omega}$ the pairing $\tilde{w}\big(L(\vec{v})\big)$ depends linearly on each argument. Now continue to the second half of the question: The first (row) index of the matrix of L, being dual to $\tilde{\omega}$, transforms like a (contravariant) vector index, and the second (column) index, being dual to the argument $\vec{v}$, transforms like a (covariant) one-form index. Thus

$$L^{\overline{\alpha}}{}_{\overline{\beta}} = \Lambda^{\overline{\alpha}}{}_{\mu} \left(\Lambda^{-1}\right)^{\nu}{}_{\overline{\beta}} \, L^{\mu}{}_{\nu} \, .$$

Starting from the concrete: Under a change of coordinates the matrix of L changes by a similarity translation, $\overline{L} = \Lambda L \Lambda^{-1}$. Written out in indices and rearranged, this equation becomes

$$L^{\overline{\alpha}}{}_{\overline{\beta}} = \Lambda^{\overline{\alpha}}{}_{\mu} \left(\Lambda^{-1}\right)^{\nu}{}_{\overline{\beta}} \, L^{\mu}{}_{\nu} \, .$$

But this is the transformation law of a $\binom{1}{1}$ tensor; any object whose components transform that way defines a $\binom{1}{1}$ tensor, regardless of whether it has an *a priori* interpretation as a bilinear mapping.

   (b) Prove that the *trace*, $L^{\alpha}{}_{\alpha}$, of a $\binom{1}{1}$ tensor is an invariant (independent of coordinate system).

Use the transformation law found above:

$$L^{\overline{\alpha}}{}_{\overline{\alpha}} = \Lambda^{\overline{\alpha}}{}_{\mu} \left(\Lambda^{-1}\right)^{\nu}{}_{\overline{\alpha}} \, L^{\mu}{}_{\nu} = \delta^{\nu}_{\mu} \, L^{\mu}{}_{\nu} = L^{\mu}{}_{\mu} \, .$$

(c) Show that the trace of a $\binom{0}{2}$ tensor is *not* an invariant. (Here the trace is again defined simply as the sum of the diagonal components, $\sum_\alpha T_{\alpha\alpha}$.)

In this case we would have

$$T_{\overline{\alpha\alpha}} = \left(\Lambda^{-1}\right)^{\mu}{}_{\overline{\alpha}} \left(\Lambda^{-1}\right)^{\nu}{}_{\overline{\alpha}} T_{\mu\nu} \,,$$

and there is no reason why this should simplify to $T_{\mu\mu}$. Indeed, suppose that $T_{\mu\nu} = \delta_{\mu\nu}$ in some frame. Then

$$T_{\overline{\alpha\alpha}} = \sum_{\mu,\alpha} \left[\left(\Lambda^{-1}\right)^{\mu}{}_{\overline{\alpha}}\right]^2,$$

which is surely not equal to $T_{\mu\mu}$ (i.e., $\sum_\mu \delta_{\mu\mu} =$ (dimension of space)) for every Λ .

(d) When $\mathcal{V}$ is Minkowski space-time, define something different that reasonably deserves to be called the trace of a $\binom{0}{2}$ tensor. Show that the result is unambiguous, even if T is not symmetric. What happens if T is antisymmetric?

Use the (inverse) metric tensor to raise an index of the $\binom{0}{2}$ tensor, thereby creating a $\binom{1}{1}$ tensor whose trace can be taken invariantly:

$$T^{\alpha}{}_{\beta} = \eta^{\alpha\gamma} T_{\gamma\beta} \,, \quad T^{\alpha}{}_{\alpha} = \eta^{\alpha\gamma} T_{\gamma\alpha} \,.$$

One might worry that the result will depend on which index is raised, if T is not symmetric. But it does not:

$$T_{\alpha\gamma}\eta^{\gamma\alpha} = \eta^{\gamma\alpha} T_{\alpha\gamma} = \eta^{\alpha\gamma} T_{\gamma\alpha}$$

(where the first step is commutativity of multiplication of numbers, and the second is just renaming of indices). If T is antisymmetric, the trace is zero because η is symmetric:

$$T^{\alpha}{}_{\alpha} = \eta^{\alpha\gamma} T_{\gamma\alpha} = -\eta^{\alpha\gamma} T_{\alpha\gamma} = -\eta^{\gamma\alpha} T_{\alpha\gamma} = -T^{\gamma}{}_{\gamma} \,.$$

2. *(45 pts.)*

(a) A particle of (rest) mass M collides with, and completely absorbs, a particle of mass m . The first particle recoils, remaining a particle of the same type. Prove algebraically (from conservation of 4-momentum) that this process is impossible! (Both M and m are positive, with $m \ll M$.)

Let $\vec{P}_i$ and $\vec{P}_f$ be the initial and final 4-momenta of the massive particle, and $\vec{p}$ the 4-momentum of the absorbed particle. Then $\vec{P}_f = \vec{P}_i + \vec{p}$. It follows that

$$-M^2 = \vec{P}_f \cdot \vec{P}_f = (\vec{P}_i + \vec{p}) \cdot (\vec{P}_i + \vec{p}) = -M^2 - m^2 + 2\vec{p} \cdot \vec{P}_i$$

so that $\vec{p} \cdot \vec{P}_i = \frac{1}{2} m^2$. In the initial rest frame of M , this equation is $-Mp^0 = \frac{1}{2} m^2$, which is inconsistent because all the masses and energies must be positive. (There are other — harder but valid — ways to reach the same conclusion.)

(b) A bullet is fired into a block of wood and lodges there. Explain physically why your argument in (a) does not apply to this scenario. (There are two important reasons! Hints are subtly embedded in the last sentence of (c).)

The outgoing body does not still have mass M . First of all, the mass of the bullet must be included. Second, not all of the kinetic energy of the bullet goes into kinetic energy of the recoiling block. Some energy is dissipated as heat or stored as potential energy of deformed wood fibers. (A nonrelativistic calculation from freshman physics shows that the naive assumption of conservation of kinetic energy is inconsistent with conservation of 3-momentum.) In relativistic mechanics this internal energy contributes to the rest mass of the recoiling body. Thus the final rest mass is *greater* than $M+m$.

(c) Make the necessary minor change in problem (a) to make it consistent (and similar to (b), to the extent that one can model blocks and bullets as "particles"). Assuming that the massive particle was initially at rest, calculate its final total energy and final speed in terms of the initial masses and the initial energy of the absorbed particle.

Let $\mathcal{M} \equiv M+m+\epsilon$ be the final rest mass. First let's write out the conservation equation $\vec{P}_f = \vec{P}_i + \vec{p}$ in the initial rest frame:

$$\mathcal{M} \begin{pmatrix} \gamma \\ \gamma\vec{v} \end{pmatrix} = \begin{pmatrix} M \\ \vec{0} \end{pmatrix} + m \begin{pmatrix} \delta \\ \delta\vec{u} \end{pmatrix},$$

where $\vec{v}$ is the final 3-velocity, and $\vec{u}$ is the initial 3-velocity of the light particle, and $\gamma = (1 - v^2)^{-1/2}$, $\delta = (1-u^2)^{-1/2}$. The first component (energy conservation) immediately gives the final energy

$$E = \mathcal{M}\gamma = M + m\delta = M + p^0.$$

One way to find the speed is to divide the momentum by the energy:

$$v = \frac{m\delta u}{M + m\delta} = \frac{up^0}{M+p^0} = \frac{\sqrt{(p^0)^2 - m^2}}{M+p^0},$$

where the last step uses $u = \sqrt{1-\delta^{-2}}$ and $\delta = p^0/m$. Another way is to find $\mathcal{M}$ first. The easiest way to do that is to go back to the calculation in (a), corrected by setting $\vec{P}_f \cdot \vec{P}_f = -\mathcal{M}^2$. The other side of the equation, evaluated in the initial rest frame, is still

$$-M^2 - m^2 + 2\vec{p}\cdot\vec{P}_f = -M^2 - m^2 - 2Mp^0.$$

Thus

$$\mathcal{M} = \sqrt{M^2 + m^2 + 2Mp^0},$$

$$\gamma = \frac{E}{\mathcal{M}} = \frac{M+p^0}{\sqrt{M^2+m^2+2Mp^0}},$$

$$v = \sqrt{1-\gamma^{-2}} = \frac{\sqrt{(p^0)^2 - m^2}}{M+p^0}.$$

3. *(Essay – 35 pts.)* Discuss the twin "paradox" and explain why it is not really a paradox, but a clear consequence of special relativity. (Diagrams will help!) Discuss at least two of these three scenarios:

   (a) The moving twin travels at constant velocity for a time t_0 (as measured by the stationary twin), then suddenly reverses course and returns home at the same speed.

   (b) The moving twin travels at constant acceleration along the parametrized space-time path

   $$t = \sinh \tau, \quad x = \cosh \tau,$$

   while the stationary twin remains at $x = x_0$. Note that the twins encounter each other when

   $$t = \pm \sinh(\cosh^{-1} x_0) = \pm\sqrt{x_0^2 - 1} \equiv \pm t_0 \,.$$

   (c) The moving twin travels at constant velocity all the way around a spatially finite (periodic) universe of length (circumference) C.