Math. 489GR (Fulling) 17 November 2005

Test B – Solutions

1. *(40 pts.)* Consider these two 2-dimensional line elements:

$$\text{(I)} \qquad ds^2 = -dt^2 + \cosh^2 t\, dx^2,$$
$$\text{(II)} \qquad ds^2 = -\cosh^2 t\, dt^2 + dx^2.$$

(a) Find the geodesic equation for case (I).

The easiest method is to write down the Lagrangian

$$L = \frac{1}{2} g_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = -\frac{1}{2}\left(\frac{dt}{ds}\right)^2 + \frac{1}{2}\cosh^2 t \left(\frac{dx}{ds}\right)^2 .$$

The the Lagrange equations are

$$0 = \frac{d}{ds}\frac{\partial L}{\partial \dot t} - \frac{\partial L}{\partial t} = -\frac{d^2 t}{ds^2} - \cosh t \sinh t \left(\frac{dx}{ds}\right)^2 ,$$
$$0 = \frac{d}{ds}\frac{\partial L}{\partial \dot x} - \frac{\partial L}{\partial x} = \frac{d}{ds} \cosh^2 t\, \frac{dx}{ds} - 0 = \cosh^2 t\, \frac{d^2 x}{ds^2} + 2\cosh t \sinh t\, \frac{dt}{dx}\, \frac{dx}{ds} .$$

These can be simplified to

$$0 = \frac{d^2 t}{ds^2} + \cosh t \sinh t \left(\frac{dx}{ds}\right)^2 ,$$
$$0 = \frac{d^2 x}{ds^2} + 2 \tanh t\, \frac{dt}{dx}\, \frac{dx}{ds} .$$

(b) A space-time manifold M has the metric tensor given by (I), with the coordinate ranges $-\infty < t < \infty$, $0 \le x < 2\pi$. (This is the two-dimensional de Sitter space.) Tell how to integrate a scalar function over M covariantly. (I.e., what is the correct "volume element" or "geometric weight factor" for these coordinates?)

The determinant of the metric tensor is

$$\begin{vmatrix} -1 & 0 \\ 0 & \cosh^2 t \end{vmatrix} = -\cosh^2 t.$$

So the integral of f over M is

$$\int_M f\,\sqrt{|g|}\, d^2 x = \int_{-\infty}^{\infty} dt \int_0^{2\pi} dx\, f(t,x)\, \cosh t\, .$$

(c) Prove that the geometry in case (II) is actually *flat*. (This is *very* easy! If you embark on a big calculation, you are missing the point.)

Introduce a new time coordinate $\tau = \sinh t$. Then $d\tau = \cosh t\, dt$, so $ds^2 = -d\tau^2 + dx^2$ and the space is revealed to be (at least locally) two-dimensional Minkowski space. ("At least locally" means that it might be just a subset of Minkowski space, or perhaps a cylinder with x a periodic coordinate. These still qualify as "flat".)

2. *(40 pts.)* Consider a metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where h is *small*. We will do calculations to first order in h.

(a) Show that $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$ to first order, but not exactly (i.e., that $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + O(h^2)$ in general).

Abstract argument: For any small (i.e., sufficiently close to the zero matrix) square matrix A,

$$(I + A)^{-1} = I - A + A^2 - A^3 + \cdots = I - A + O(A^2).$$

Here I is the identity (unit) matrix. But we need to replace I by $\eta = \mathrm{diag}(-1, 1, 1, 1)$. Write $\eta + h = \eta(I + \eta^{-1}h)$ so that

$$(\eta + h)^{-1} = (I + \eta^{-1}h)^{-1}\eta^{-1} = (I - \eta^{-1}h + O(h^2))\eta^{-1} = \eta^{-1} - \eta^{-1}h\eta^{-1} + O(h^2).$$

Since η^{-1} is just η with the indices up, this is what we want to prove.

Concrete argument: Write out

$$(\eta_{\mu\rho} + h_{\mu\rho})(\eta^{\rho\nu} - h^{\rho\nu}) = \delta^\nu_\mu + h_\mu{}^\nu - h_\mu{}^\nu - h_{\mu\rho}h^{\rho\nu} = \delta^\nu_\mu + O(h^2).$$

So the formula for the inverse is almost right but not quite. To kill off the $O(h^2)$ term in the last equation we could add a certain $O(h^2)$ term to the formula for $g^{\rho\nu}$, and so on. (To make the conclusion rigorous: The difference between the true inverse and the approximation, when multiplied by $\eta + h$, is of order $O(h^2)$. Since $\eta + h$ is nonsingular (for "small" h), it follows that the difference (error) is itself of order $O(h^2)$.)

(b) Calculate the Christoffel symbols to first order in h.

Let us use the formula

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}\, g^{\mu\gamma}\left(g_{\gamma\beta,\alpha} + g_{\alpha\gamma,\beta} - g_{\alpha\beta,\gamma}\right).$$

(Using the geodesic Lagrangian does not help much in this case (but it works).) The derivatives are already first-order in h. Therefore, we can throw away the h term in $g^{\mu\gamma}$, because it will make only second-order contributions. Thus

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}\, \eta^{\mu\gamma}\left(h_{\gamma\beta,\alpha} + h_{\alpha\gamma,\beta} - h_{\alpha\beta,\gamma}\right).$$

(c) Knowing that

$$R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha_{\beta\nu,\mu} - \Gamma^\alpha_{\beta\mu,\nu} + \Gamma^\alpha_{\sigma\mu}\Gamma^\sigma_{\beta\nu} - \Gamma^\alpha_{\sigma\nu}\Gamma^\sigma_{\beta\mu}\,,$$

calculate $R^\alpha{}_{\beta\mu\nu}$ to first order in h. (The answer has four terms, two with plus signs and two with minus signs.)

Since $\Gamma = O(h)$, the $\Gamma\Gamma$ terms are $O(h^2)$ and can be ignored. Thus

$$\begin{aligned} R^\alpha{}_{\beta\mu\nu} &= \frac{1}{2}\eta^{\alpha\gamma}(h_{\gamma\nu,\beta\mu} + h_{\beta\gamma,\nu\mu} - h_{\beta\nu,\gamma\mu}) - \frac{1}{2}\eta^{\alpha\gamma}(h_{\gamma\mu,\beta\nu} + h_{\beta\gamma,\mu\nu} - h_{\beta\mu,\gamma\nu}) \\ &= \frac{1}{2}\eta^{\alpha\gamma}(h_{\gamma\nu,\beta\mu} - h_{\gamma\mu,\beta\nu} - h_{\beta\nu,\gamma\mu} + h_{\beta\mu,\gamma\nu}). \end{aligned}$$

This agrees with Schutz (8.25).

3. *(45 pts.)* In Euclidean $\mathbf{R}^2$ introduce *parabolic coordinates* (u,v) by

$$x = \tfrac{1}{2}(u^2 - v^2), \qquad y = uv.$$

Note that the tangent vectors to the coordinate curves,

$$\vec{e}_u = \frac{\partial \mathbf{r}}{\partial u} = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad \vec{e}_v = \frac{\partial \mathbf{r}}{\partial v} = \begin{pmatrix} -v \\ u \end{pmatrix} \qquad (\text{where } \mathbf{r} = \begin{pmatrix} x \\ y \end{pmatrix})$$

form a local basis at each point $\mathbf{r}$ for the contravariant vectors. (This basis happens to be orthogonal but not orthonormal.)

(a) Are there any points where the assertion above ("Note that … ") is not true?

At the origin, where $u = v = 0$ as well as $x = y = 0$, the two tangent vectors are both zero and hence do not form a basis. At any other point the vectors are linearly independent (their determinant is $u^2 + v^2 \neq 0$) and orthogonal, so the assertion is true. (When $u^2 + v^2 = 1$ they are of unit length and hence the basis is actually orthonormal.)

(b) *(essay)* Using this coordinate system as an example, explain how *covariant derivatives* and *Christoffel symbols* in a flat space are related to local basis vectors.

In flat space the Cartesian basis vectors are regarded as the same at all points, and so we can differentiate vector fields simply by differentiating their Cartesian components. When a curvilinear coordinate system is used, the basis vectors change from point to point, and hence the derivative of a vector field includes contributions from the derivatives of the basis vectors:

$$\nabla_\mu(v^\alpha \vec{e}_\alpha) = v^\alpha{}_{,\mu}\vec{e}_\alpha + v^\alpha \nabla_\mu \vec{e}_\alpha.$$

Therefore, in the notation

$$v^\beta{}_{;\mu} = v^\beta{}_{,\mu} + \Gamma^\beta_{\alpha\mu} v^\alpha$$

we can identify

$$\Gamma^\beta_{\alpha\mu} = (\nabla_\mu \vec{e}_\alpha)^\beta$$

(the β component of the expansion of $\nabla_\mu \vec{e}_\alpha$ as a linear combination of the curvilinear basis vectors).

(c) Calculate the Christoffel symbols for the parabolic coordinate system, either by the method suggested by your essay or by another method. (Of course, if you have time you will do it by two methods to check your answers and get a bit of extra credit.)

Basis-vector method: We have immediately

$$\vec{e}_{u,u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{e}_{u,v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \vec{e}_{v,u} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \vec{e}_{v,v} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

Unfortunately, the components of these vectors are with respect to the Cartesian basis, so we need to express the latter in terms of the parabolic one. From

$$\vec{e}_u = u\vec{e}_x + v\vec{e}_y\,, \quad \vec{e}_v = -v\vec{e}_x + u\vec{e}_y$$

we find

$$u\vec{e}_u - v\vec{e}_v = (u^2+v^2)\vec{e}_x\,, \quad v\vec{e}_u + u\vec{e}_v = (u^2+v^2)\vec{e}_y\,,$$

hence

$$\vec{e}_{u,u} = \vec{e}_x = \frac{u}{u^2+v^2}\,\vec{e}_u - \frac{v}{u^2+v^2}\,\vec{e}_v = -\vec{e}_{v,v}\,, \qquad \vec{e}_{u,v} = \vec{e}_{v,u} = \vec{e}_y = \frac{v}{u^2+v^2}\,\vec{e}_u + \frac{u}{u^2+v^2}\,\vec{e}_v\,.$$

Therefore,

$$\Gamma^u_{uu} = \frac{u}{u^2+v^2}\,, \quad \Gamma^v_{uu} = \frac{-v}{u^2+v^2}\,, \quad \Gamma^u_{vv} = \frac{-u}{u^2+v^2}\,, \quad \Gamma^v_{vv} = \frac{v}{u^2+v^2}\,,$$

$$\Gamma^u_{uv} = \Gamma^u_{vu} = \frac{v}{u^2+v^2}\,, \quad \Gamma^v_{uv} = \Gamma^v_{vu} = \frac{u}{u^2+v^2}\,.$$

Metric methods: First we need to find the metric in these coordinates. One way is to recall that $g_{\alpha\beta} = \vec{e}_\alpha \cdot \vec{e}_\beta$, so

$$\begin{pmatrix} g_{uu} & g_{uv} \\ g_{vu} & g_{vv} \end{pmatrix} = \begin{pmatrix} u^2+v^2 & 0 \\ 0 & u^2+v^2 \end{pmatrix}.$$

Another is to crank out

$$ds^2 = dx^2 + dy^2 = (u\,du - v\,dv)^2 + (v\,du + u\,dv)^2 = (u^2+v^2)(du^2+dv^2).$$

Now we can either use the gamma formula recalled in Qu. 2 or use the geodesic Lagrangian. I will do the latter:

$$L = \tfrac{1}{2}(u^2+v^2)(\dot{u}^2+\dot{v}^2),$$

so (looking at u first)

$$0 = \frac{d}{ds}\frac{\partial L}{\partial \dot{u}} - \frac{\partial L}{\partial u} = \ddot{u}(u^2+v^2) + 2u\dot{u}^2 + 2v\dot{u}\dot{v} - u(\dot{u}^2+\dot{v}^2),$$

or

$$0 = \ddot{u} + \frac{u}{u^2+v^2}\,(\dot{u}^2 - \dot{v}^2) + \frac{2v}{u^2+v^2}\,\dot{u}\dot{v}.$$

From this we read off

$$\Gamma^u_{uu} = \frac{u}{u^2+v^2}\,, \quad \Gamma^u_{vv} = \frac{-u}{u^2+v^2}\,, \quad \Gamma^u_{uv} = \frac{v}{u^2+v^2}\,.$$

Since L is symmetrical in u and v, it is obvious that the v equations will be exactly like the u equations except that u and v are interchanged.