On the Relation between Inversion and Index Swapping

In special relativity, Schutz writes \( \{\Lambda^\beta_\alpha\} \) for the matrix of the coordinate transformation inverse to the coordinate transformation

\[ x^{\bar{\alpha}} = \Lambda^\bar{\alpha}_\beta x^\beta. \]  

However, one might want to use that same notation for the transpose of the matrix obtained by raising and lowering the indices of the matrix in (*):

\[ \Lambda_{\bar{\alpha}}^\beta = g_{\bar{\alpha}\bar{\mu}} \Lambda^{\bar{\mu}}_{\nu} g^{\nu\beta}. \]

Here \( \{g_{\alpha\beta}\} \) and \( \{g_{\bar{\alpha}\bar{\beta}}\} \) are the matrices of the metric of Minkowski space with respect to the unbarred and barred coordinate system, respectively. (The coordinate transformation (*) is linear, but not necessarily a Lorentz transformation.) Let us investigate whether these two interpretations of the symbol \( \Lambda^\beta_\alpha \) are consistent.

If the answer is yes, then (according to the first definition) \( \delta_{\bar{\gamma}}^{\bar{\alpha}} \) must equal

\[
\begin{align*}
\Lambda^{\bar{\alpha}}_\beta \Lambda_{\bar{\gamma}}^\beta &\equiv \Lambda^{\bar{\alpha}}_\beta (g_{\bar{\gamma}\bar{\mu}} \Lambda^{\bar{\mu}}_{\nu} g^{\nu\beta}) \\
&= g_{\bar{\gamma}\bar{\mu}} (\Lambda^{\bar{\mu}}_{\nu} g^{\nu\beta} \Lambda_{\bar{\alpha}}^\beta) \\
&= g_{\bar{\gamma}\bar{\mu}} g^{\mu\bar{\alpha}} \\
&= \delta_{\bar{\gamma}}^{\bar{\alpha}}, \quad \text{Q.E.D.}
\end{align*}
\]

(The first step uses the second definition, and the next-to-last step uses the transformation law of a \((\frac{2}{0})\) tensor.)

In less ambiguous notation, what we have proved is that

\[
(\Lambda^{-1})^{\beta}_{\bar{\alpha}} = g_{\alpha\bar{\mu}} \Lambda^{\bar{\mu}}_{\nu} g^{\nu\beta}. \tag{†}
\]

Note that if \( \Lambda \) is not a Lorentz transformation, then the barred and unbarred \( g \) matrices are not numerically equal; at most one of them in that case has the form

\[
\eta = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

If \( \Lambda \) is Lorentz (so that the \( g \) matrices are the same) and the coordinates are with respect to an orthogonal basis (so that indeed \( g = \eta \)), then (†) is the indefinite-metric counterpart of the “inverse = transpose” characterization of an orthogonal matrix in Euclidean space: The inverse of a Lorentz transformation equals the transpose with the indices raised and lowered (by \( \eta \)). (In the Euclidean case, \( \eta \) is replaced by \( \delta \) and hence (†) reduces to

\[
(\Lambda^{-1})^{\beta}_{\bar{\alpha}} = \Lambda^{\bar{\alpha}}_\beta,
\]
in which the up-down index position has no significance.) For a general linear transformation, \((\dagger)\) may appear to offer a free lunch: How can we calculate an inverse matrix without the hard work of evaluating Cramer’s rule, or performing a Gaussian elimination? The answer is that in the general case at least one of the matrices \(\{g_{\bar{\alpha}\bar{\mu}}\}\) and \(\{g^{\nu\beta}\}\) is nontrivial and somehow contains the information about the inverse matrix.

**Alternative argument:** We can use the metric to map between vectors and covectors. Since

\[ v^{\bar{\alpha}} = \Lambda_{\bar{\alpha}}^{\bar{\beta}} v^{\bar{\beta}} \]

is the transformation law for vectors, that for covectors must be

\[ \tilde{v}_{\bar{\mu}} = g_{\bar{\mu}\bar{\alpha}} v^{\bar{\alpha}} \]

\[ = g_{\bar{\mu}\bar{\alpha}} \Lambda_{\bar{\alpha}}^{\bar{\beta}} v^{\bar{\beta}} \]

\[ = g_{\bar{\mu}\bar{\alpha}} \Lambda_{\bar{\alpha}}^{\bar{\beta}} g^{\bar{\beta}\nu} \tilde{v}_{\nu} \]

\[ \equiv \Lambda_{\bar{\mu}}^{\nu} \tilde{v}_{\nu} \]

according to the second definition. But the transformation matrix for covectors is the transpose of the inverse of that for vectors — i.e.,

\[ \tilde{v}_{\bar{\mu}} = \Lambda_{\bar{\mu}}^{\nu} \tilde{v}_{\nu} \]

according to the first definition. Therefore, the definitions are consistent.