Solutions to Exercises 2.19 and 2.21

2.19(a), Method 1a (credit John Cesar and others): We are given that \( \vec{a} \) has constant spatial direction, so its spatial components at any time are the same as at any other time except possibly for an overall multiplicative constant:

\[
\vec{a}(t_1) = (a^0, a^1, a^2, a^3), \quad \vec{a}(t_2) = (b^0, Ca^1, Ca^2, Ca^3).
\]

According to (2.32), \( \vec{U} \cdot \vec{a} = 0 \); it follows that in the body’s MCRF, \( a^0 = 0 = b^0 \). But we also are given that \( \vec{a} \) always has the same magnitude, so we get

\[
a_1^2 + a_2^2 + a_3^2 = C^2(a_1^2 + a_2^2 + a_3^2).
\]

Thus \( |C| = 1 \), and to preserve direction \( C = +1 \). Thus at all times

\[
\vec{a}(t) = (0, a^1, a^2, a^3) \quad \text{in MCRF},
\]

and its spatial part is the Galilean acceleration.

Method 1b (credit Sean Grant and others):

\[
\vec{U} = \frac{d\vec{x}}{d\tau} = \gamma \begin{pmatrix} 1, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \end{pmatrix},
\]

\[
\vec{a} = \frac{d\vec{U}}{d\tau} = \gamma \left[ \frac{d\gamma}{dt} \begin{pmatrix} 1, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \end{pmatrix} + \gamma \begin{pmatrix} 0, \frac{d^2x}{dt^2}, \frac{d^2x}{dt^2}, \frac{d^2x}{dt^2} \end{pmatrix} \right]
\]

\[
= \gamma \begin{pmatrix} \frac{d\gamma}{dt} \begin{pmatrix} 1 \end{pmatrix} + \gamma \begin{pmatrix} 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix} + \gamma \begin{pmatrix} a \end{pmatrix},
\]

where \( a \) is the Galilean acceleration. Furthermore,

\[
\gamma = (1 - v^2)^{-1/2} \Rightarrow \frac{d\gamma}{dt} = v \frac{dv}{dt} (1 - v^2)^{-3/2}.
\]

If we are in the MCRF, \( \gamma = 1 \) and \( v = 0 \) and hence \( \frac{d\gamma}{dt} = 0 \). So we end up with

\[
\vec{a} = \begin{pmatrix} 0 \\ a \end{pmatrix} \quad \text{in MCRF}.
\]

Method 2: Do 2.21 first.

\[
\frac{d}{d\lambda} \begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \cosh(\lambda/a) \\ \sinh(\lambda/a) \end{pmatrix} \equiv \vec{U}.
\]

Note that \( \vec{U} \cdot \vec{U} = -1 \), so we can identify \( \vec{U} \) as the 4-velocity and \( \lambda \) as the proper time. Then

\[
\frac{d}{d\tau} \vec{U} = \frac{d^2}{d\lambda^2} \begin{pmatrix} t \\ x \end{pmatrix} = \frac{1}{a} \begin{pmatrix} \sinh(\tau/a) \\ \cosh(\tau/a) \end{pmatrix},
\]
which has constant magnitude $a^{-2}$ (and constant spatial direction, since we’re considering only one spatial dimension). Thus this worldline has uniform acceleration, $\alpha = 1/a$, The Lorentz transformation into the MCRF (mapping $\vec{U}$ to $(1,0)$) is

$$\Lambda = \begin{pmatrix} \cosh(\lambda/a) & -\sinh(\lambda/a) \\ -\sinh(\lambda/a) & \cosh(\lambda/a) \end{pmatrix}. \tag{1}$$

Applied to $\vec{a}$, $\Lambda$ yields

$$\frac{1}{a} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha \end{pmatrix},$$

which again proves 2.19(a). (Clearly this worldline can be embedded into 4-dimensional space-time, the two extra dimensions remaining inert. Any other uniform-acceleration worldline can be put into this form by rotation and translation of coordinates.)

2.19(b), Method 1: Use Ex. 2.21. Without loss of generality we can write

$$t = \frac{1}{\alpha} \sinh(\alpha \tau), \quad x = \frac{1}{\alpha} \cosh(\alpha \tau),$$

$$\vec{U} = \begin{pmatrix} \cosh(\alpha \tau) \\ \sinh(\alpha \tau) \end{pmatrix} = \begin{pmatrix} \gamma \\ \gamma v \end{pmatrix} \Rightarrow v = \frac{\sinh(\alpha \tau)}{\cosh(\alpha \tau)} = \tanh(\alpha \tau).$$

Solving, we have

$$\tau = \frac{1}{\alpha} \sinh^{-1}(\alpha t)$$

and hence

$$v = \tanh(\sinh^{-1}(\alpha t)). \tag{2}$$

We can get rid of the hyperbolic functions:

$$v = \frac{\sinh(\sinh^{-1}(\alpha t))}{\cosh(\sinh^{-1}(\alpha t))}$$

$$= \frac{\alpha t}{\sqrt{1 + (\alpha t)^2}}. \tag{3}$$

The inverse equation inverse to (3) (needed to answer the numerical question) is

$$t = \frac{1}{\alpha} \frac{v}{\sqrt{1 - v^2}}.$$ 

The change in $x$ is

$$\Delta x = \frac{1}{\alpha} \cosh(\alpha \tau) - \frac{1}{\alpha}$$

(note that $x(0) = 1/\alpha$, not 0), so

$$\Delta x = \frac{1}{\alpha} \left[ \cosh(\sinh^{-1}(\alpha t)) - 1 \right] = \frac{1}{\alpha} \left[ \sqrt{1 + (\alpha t)^2} - 1 \right].$$
For $\alpha = 10 \text{ m/s}^2 = \frac{1}{7} \times 10^{-7} \text{ s}^{-1}$ and $v = 0.999$, one gets $t \approx \frac{2}{3} \times 10^9 \text{ s} \approx 20 \text{ years}$.

(A convenient fact to remember is that 1 year $\approx \pi \times 10^7$ seconds.) (Numerical answers on student papers were all over the lot, even among those who had the correct formulas.)

Method 2(a) (credit Robert DeAlba and others): (I suppress the two transverse dimensions.) Transform

$\vec{U} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{a} = \begin{pmatrix} 0 \\ \alpha \end{pmatrix}$

from the MCRF back to the inertial frame by [in notation set by (1)]

$\Lambda^{-1} = \begin{pmatrix} \cosh(\lambda/a) & \sinh(\lambda/a) \\ \sinh(\lambda/a) & \cosh(\lambda/a) \end{pmatrix}$,

getting

$\vec{U} = \begin{pmatrix} \gamma \\ \gamma v \end{pmatrix}, \quad \begin{pmatrix} v\gamma \alpha \\ \gamma \alpha \end{pmatrix} = \vec{a} = \frac{d\vec{U}}{d\tau}$.

Thus

$\frac{d\gamma}{d\tau} = v\gamma \alpha, \quad \frac{d}{d\tau}(\gamma v) = \gamma \alpha$.

From either of these equations and the definition of $\gamma$ you get after several steps of calculus and algebra

$\alpha = \gamma^2 \frac{dv}{d\tau}$.

Therefore,

$\alpha \int d\tau = \int \gamma^2 dv = \int \frac{dv}{1 - v^2}$,

$\alpha \tau = \tanh^{-1} v \Rightarrow v = \tanh(\alpha \tau)$.  \hspace{1cm} (4)

Since

$v = \frac{dx}{dt} = \frac{dx/d\tau}{dt/d\tau} \equiv \frac{U_1}{U_0}$

and $\vec{U}$ is normalized to $-1$, it follows that

$\vec{U} = \begin{pmatrix} \cosh(\alpha \tau) \\ \sinh(\alpha \tau) \end{pmatrix}$.

Hence

$\frac{dt}{d\tau} = U_0 = \cosh(\alpha \tau) \Rightarrow t = \frac{1}{\alpha} \sinh(\alpha \tau)$,

so (2) follows from (4) and you can proceed as in Method 1.

There were several variations on this theme that somehow led directly to (3) without going through (2).
2.19(c): In either approach above, we already know the relations among \( \tau, t, \) and \( x. \)

From \( \tau = \alpha^{-1} \sinh^{-1}(\alpha t) \) and the numerical values of \( \alpha \) and \( t \) above, we get \( \tau = 1.14 \times 10^8 \) s = 3.6 years. Then from either formula for \( \Delta x \) above we get \( \Delta x = 6.37 \times 10^8 \) s = 1.9 \times 10^{17} \) m.

For the center of the galaxy, we have \( \Delta x = 6.7 \times 10^{11} \) s and

\[
\tau = \frac{1}{\alpha} \cosh^{-1}(1 + \alpha \Delta x) = 3.2 \times 10^8 \text{ s} \approx 10 \text{ years}.
\]