Consider $z^3 + a_1 z + a_0 = 0$, where a_1 and a_0 are real. Let

$$q = \frac{a_1}{3}\,, \qquad r = -\frac{a_0}{2}\,.$$

Then (according to the handbook)

- If $q^3 + r^2 > 0$, there are one real root and a pair of complex conjugate roots,
- If $q^3 + r^2 = 0$, all roots are real and at least two are equal,
- If $q^3 + r^2 < 0$, all roots are real.

In Bombelli's case, $a_1 = -8$ and $a_0 = -3$, so

$$q = -\frac{8}{3}\,, \qquad r = \frac{3}{2}\,, \qquad q^3 + r^2 = -\frac{1805}{108} = -\frac{5}{3}\left(\frac{19}{6}\right)^2.$$

In the trisection case, $a_1 = -3$ and $a_0 = -1$, so

$$q = -1, \qquad r = \frac{1}{2}\,, \qquad q^3 + r^2 = -\frac{3}{4}\,.$$

In both cases all the roots are real. (Recall that that is when the imaginary numbers in the formula are the greatest embarrassment.)

The handbook goes on to define s_1 and s_2 as

$$\sqrt[3]{r \pm \sqrt{q^3 + r^2}}$$

and list the roots of the cubic as

$$\begin{aligned}
z_1 &= s_1 + s_2\,,\\
z_2 &= -\frac{1}{2}(s_1 + s_2) + \frac{i\sqrt{3}}{2}(s_1 - s_2),\\
z_3 &= -\frac{1}{2}(s_1 + s_2) - \frac{i\sqrt{3}}{2}(s_1 - s_2),
\end{aligned}$$

and lets it go at that. However, since a complex number usually has two different square roots and three different cube roots, one needs to look at these formulas critically to get all the right roots and avoid redundancy. In the case when all roots are real (which is all I'll discuss) we see that ${s_1}^3$ and ${s_2}^3$ form a complex conjugate pair and the ambiguity in the sign of the square root is simply the freedom to interchange the two. What is less obvious is that the ambiguity in the choice of the cube roots amounts to permuting the solutions z_j. Let's come back to that issue later.

In the Cardano–Bombelli problem we see that z_1 is equivalent to Cardano's formula as Bombelli interpreted it.

For the trisection of 60° we have

$${s_1}^3 = \frac{1}{2} + i\,\frac{\sqrt{3}}{2}\,, \qquad {s_2}^3 = \frac{1}{2} - i\,\frac{\sqrt{3}}{2}\,.$$

One may recognize these as negatives of the two nonreal cube roots of unity, hence they are cube roots of -1. Therefore, s_1 and s_2 are ninth roots of -1. Constructing the value of $\sqrt[9]{-1}$ with smallest argument (i.e., closest to a positive real number and in first quadrant),

$$s_1 = \cos\frac{2\pi}{9} + i\,\sin\frac{2\pi}{9}\,,$$

is tantamount to trisecting the angle $2\pi/3 = 60°$. This verifies that we are on the right track, albeit a circular one! We now get

$$\begin{aligned} z_1 &= 2\cos\frac{2\pi}{9}\,, \\ z_{2,3} &= -\cos\frac{2\pi}{9} \mp \sqrt{3}\,\sin\frac{2\pi}{9}\,. \end{aligned}$$

The first of these roots is the one that gives the answer to the trisection problem.

Now, what about those other two roots? Suppose we define $s_1 = a + ib$ to be *any* of the three possible cube roots of the known ${s_1}^3$ and take $s_2 = a - ib$. Then

$$z_1 = 2a, \qquad z_{2,3} = -a \mp \sqrt{3}\,b.$$

So far, so good. (Note that all the roots are real and that their sum is 0, which is the negative of the coefficient of the quadratic term in the cubic equation, as must be the case.) Now suppose we had chosen one of the other two cube roots,

$$s_1 = (a+ib)e^{\pm 2\pi i/3}.$$

Consider the case $\pm = +$ and work it out:

$$s_1 = (a+ib)\left(\cos\frac{2\pi}{3} + i\,\sin\frac{2\pi}{3}\right) = \left(-\frac{a}{2} - \frac{b\sqrt{3}}{2}\right) + i\left(-\frac{b}{2} + \frac{a\sqrt{3}}{2}\right).$$

If s_2 is the conjugate of this (which you get by rotating the old s_2 in the reverse direction), we see from the formulas for the roots that

$$\begin{aligned} z_1 &= -a - b\sqrt{3} = \text{old } z_2\,, \\ z_2 &= -a + b\sqrt{3} = \text{old } z_3\,, \\ z_3 &= 2a = \text{old } z_1\,. \end{aligned}$$

In this sense the three roots of the cubic do correspond to the three different ways of defining the cube root in the basic formula.