Reduction of order
(help on p. 337 and Exercise 12.10)

Euler started with the equation

$$a^3 d^3 y = y\,dx^3 \tag{1}$$

and observed that one solution is $e^{x/a}$ (or any constant multiple thereof, since the equation is linear and homogeneous). He divided by that solution and guessed that that differential expression was indeed an exact differential — more precisely, that

$$e^{-x/a}(a^3\,d^3 y - y\,dx^3) = d[e^{-x/a}(A\,d^2 y + B\,dy\,dx + Cy\,dx^2)] \tag{2}$$

for some constants A, B, C. If you calculate the differential on the right side of (2) you get four types of terms, and hence a sufficient condition for (2) to hold is a certain system of 4 equations in 3 unknowns. Miraculously, the 4 equations are not independent, and the system has a unique solution. As a result the right side of (2) comes out to be

$$a\,d[e^{-x/a}\omega], \quad \text{where} \quad \omega = a^2\,d^2 y + a\,dy\,dx + y\,dx^2. \tag{3}$$

So far we have not used the assumption that y is a solution of (1). When we do, we can conclude that $d[e^{-x/a}\omega] = 0$. This does not authorize you to conclude immediately that $\omega = 0$ (which is our goal). All you can say so far is that

$$a^2\,d^2 y + a\,dy\,dx + y\,dx^2 = \omega = Ke^{x/a} \tag{4}$$

for some constant K. However, you can compute that the known solution $e^{x/a}$ satisfies (4) for a particular value of K (which I leave you to compute). We are interested in *other* solutions, *linearly independent* of that one. Given any such solution, you can subtract some multiple of $e^{x/a}$ to get another (nonzero) solution that satisfies (4) with $K = 0$. (Here it is crucial that (4) is a *linear* equation.) So, once Euler found the general solution of $\omega = 0$, he knew he could get all the solutions of (1) by adding on arbitrary multiples of $e^{x/a}$ to arbitrary solutions of $\omega = 0$.