Exact differential equations
(help on p. 341 and Exercise 12.15)

We are supposed to consider a differential equation in the form

$$P\,dx + Q\,dy = 0. \tag{1}$$

Let's consider the example

$$P\,dx + Q\,dy = e^y\,dx + (xe^y + 1)\,dy = 0. \tag{2}$$

We calculate $\frac{\partial P}{\partial y} = e^y$ and $\frac{\partial Q}{\partial x} = e^y$. In particular,

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}\,, \tag{3}$$

so by Clairaut's theorem, there is a function $f(x, y)$ such that

$$P\,dx + Q\,dy = df. \tag{4}$$

A differential equation for which this happy accident occurs is called *exact*.

[REMARK: Clairaut's theorem is valid only if $P(x, y)$ and $Q(x, y)$ are given (and nonsingular) throughout a "simply connected" region in the plane. The canonical counterexample is

$$\frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = \text{“}d\theta\text{”},$$

for which

$$\frac{\partial P}{\partial y} = \frac{y^2 - x^2}{(x^2+y^2)^2} = \frac{\partial Q}{\partial x}$$

except at the origin, but θ, if it exists, must increase by 2π along any path encircling the origin counterclockwise — thus must be discontinuous along some curve joining the origin to infinity, contradicting the hypothesis (4) that it has P and Q as partial derivatives everywhere except the origin. Indeed, if one defines plane polar coordinates by

$$x = r\,\cos\theta, \quad y = r\,\sin\theta,$$

it is easy to see that "locally",

$$\theta = \tan^{-1}\frac{y}{x} + \text{constant},$$

which has those partial derivatives, but there is no way to choose the constant so that θ is continuously defined "globally": there must be a jump by 2π somewhere. In modern terminology, a differential expression $P\,dx + Q\,dy$ satisfying (3) is called "closed", whereas one satisfying (4) is called "exact". Exact implies closed, but closed implies exact only locally.]

Now we proceed to find f in the case (2), where putatively

$$df = e^y\,dx + (xe^y + 1)\,dy = 0.$$

Integrating with respect to x with y fixed, we get

$$f(x,y) = xe^y + r(y), \tag{5}$$

where r for the moment is an unknown function. [Note that in this calculation, *unlike everywhere else in the discussion of ordinary differential equations*, we must think of x and y as *independent* variables, not related (at the moment) by being coordinates of points on a particular curve that solves the differential equation.] Differentiate (5) with respect to y:

$$xe^y + 1 = Q = \frac{\partial f}{\partial y} = xe^y + r'(y).$$

(Presumably what Katz means by "an ordinary differential equation in one variable" is the resulting condition $r' = 1$.) It follows that $r(y) = y + \text{constant}$, and hence

$$f(x,y) = xe^y + y$$

is a function satisfying (4).

Why do we care? Well, what this calculation shows is that if we have a curve satisfying (2), or more generally (1), then $f(x,y)$ does not change as we move along that curve. That is, the curve is a locus

$$f(x,y) = K \tag{6}$$

for some arbitrary but fixed constant K. (In modern language, we would need an initial condition to determine K.) In our example, $xe^y + y = K$; unfortunately, there is no solution for y in terms of elementary functions of x in this case. Remember, however, that for Clairaut and his predecessors, the main interest was in the curve, not the formula for y, and they would probably have been satisfied with (6) as a specification of the locus.

What happens if the happy accident does not occur? Indeed, a modern student would probably write (2) as

$$\frac{dy}{dx} = -\frac{e^y}{xe^y + 1}\,, \quad \text{hence} \quad \frac{e^y}{xe^y + 1}\,dx + dy = 0, \tag{7}$$

and be stumped. If you are lucky, you may find a function $I(x,y)$ such that your differential expression becomes an exact differential after you multiply it by I. In case (7) we know that $I = xe^y + 1$ will work (because I constructed the problem that way). A more useful example is the general first-order linear equation $y' + Py = Q$, for which the integrating factor $I = e^{\int P\,dx}$ leads to the solution at the bottom of p. 353 of Katz. It is noteworthy that I is the reciprocal of a solution of the corresponding homogeneous equation, $y' + Py = 0$, which may be found by elementary means. Euler's third-order equation, to which we turn next, provides a much more subtle example of an integrating factor that is the reciprocal of a previously known solution — in that case, of the equation itself.